J. Fluid Mech. (1996), vol. 319, pp. 323–352

Wave drift damping of floating bodies in slow yaw motion

By JOHN GRUE AND ENOK PALM

Mechanics Division, Department of Mathematics, University of Oslo, Norway

(Received 2 May 1995 and in revised form 8 December 1995)

Wave diffraction, wave forces and wave drift damping due to a floating body performing a slow rotation about the vertical axis (yaw) is considered. The rotation angle of the body may be arbitrary. The angular velocity is assumed small compared to the wave frequency, however. The problem is formulated in the frame of reference following the slow rotation of the body, accounting for non-Newtonian forces. By applying the method of multiple timescales, the fluid flow is determined consistently to leading order in the slow angular velocity and to second order in the wave amplitude. Mathematical solution of the problem is obtained by means of integral equations that are applicable to geometries of arbitrary shape. The wave loads are found by applying conservation of linear and angular momentum. The wave drift damping is expressed by the far-field amplitudes of the wave field and the dipole moments of the time-averaged second-order potential. Numerical results are presented for a ship and a vertical cylinder describing a circular path in the horizontal plane. The results show that the wave drift damping due to a slow yaw motion of a floating body is one order of magnitude larger than the time-averaged forces and moment when there is no rotation. Wave drift damping due to slow rotation and slow translation are found to be of equal importance.

1. Introduction

The induced forces on and motions of floating bodies in ocean waves are topics of considerable interest both from a practical and fundamental point of view. Within deep-sea technology, for example, new floating production systems like moored ships and small floating oil platforms are under development for operations in very deep water. The actual water depths are as large as 1000–1500 m, which applies to the oil resources at Vøringsbassenget and Mørebassenget in the Norwegian Sea, but water depths down to 2000 m are also considered. Accurate computations of wave loads on and wave-induced motions of the floating parts of the production systems are crucial for the construction and dimensioning of the mooring system, and for the positioning and operation of the whole structure. There are several other examples relating to offshore activity: towing operations of large bodies, manoeuvering of ships, motions of a body drifting in waves.

While the linear part of the wave forces oscillates with the frequencies of the incoming waves, nonlinear effects give rise to sum- and difference-frequency forces acting on the moored body. The difference-frequency forces may give rise to resonant slowly varying oscillations of the body in the horizontal plane, which may have quite large amplitudes, being determined by the difference-frequency loading, the mass-spring characteristics of the body and the moorings, and by the damping forces. We shall in this contribution study wave drift damping, which has proved to be an important damping force of such resonant slowly varying motions, where damping due to linear wave radiation is negligible and viscous forces may be small. Wave drift damping is proportional to the square of the incoming wave amplitude and proportional to the slowly varying velocity of the body.

If the body performs a slow horizontal translation with speed U in the incoming waves, the time-averaged force along the speed direction, F_x, being proportional to the wave amplitude squared, is a function of the forward speed. If U is small compared to the phase velocity g/ω of the incoming waves, assuming deep water, where ω denotes the wave frequency measured in an absolute frame of reference, and g the acceleration due to gravity, the force may be expanded as

$$F_x(U) = F_{x0} - B_{11}U\omega/g. \tag{1.1}$$

Here, F_{x0} denotes the force for $U = 0$, and $-B_{11}U\omega/g$ the wave drift damping force. The expansion (1.1) was first suggested by Wichers & van Sluijs (1979), who studied model tests of the damping of low-frequency oscillations of moored ships, finding a pronounced effect of the wave drift damping. For a body translating slowly horizontally with speed U along the x-axis, speed V along the y-axis, and a slow rotation with angular velocity Ω about the vertical z-axis, with time-averaged horizontal force components F_x and F_y along the x- and y-axes, respectively, and time-averaged moment about the vertical axis, M_z, in a Cartesian frame of reference $Oxyz$, the generalization of (1.1) reads

$$\begin{pmatrix} F_x \\ F_y \\ M_z \end{pmatrix} = \begin{pmatrix} F_{x0} \\ F_{y0} \\ M_{z0} \end{pmatrix} - \begin{pmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{pmatrix} \begin{pmatrix} U\omega/g \\ V\omega/g \\ \Omega/\omega \end{pmatrix}. \tag{1.2}$$

Here, $(F_{x0}, F_{y0}, M_{z0}) = (F_x, F_y, M_z)$ for $U = V = \Omega = 0$, and $\{B_{ij}\}$ denotes the wave drift damping matrix. In recent years several methods have been published for predicting the forces due to a body moving in translatory motion in waves, or a stationary body in waves and a current, see e.g. Grue & Palm (1985, 1986) for the two-dimensional case, and in three dimensions Huismans & Hermans (1985), Huismans (1986), Zhao *et al* (1988), Zhao & Faltinsen (1989), Nossen, Grue & Palm (1991), Emmerhoff & Sclavounos (1992), Grue & Biberg (1993). These works describe methods to evaluate, among others, the wave drift damping coefficients B_{11}, B_{12}, B_{21}, B_{22}.

The theory was extended by Grue & Palm (1993) to account for the time-averaged moment M_z due to a translating body, allowing predictions of B_{61} and B_{62}. They pointed out that time-averaged velocities in the fluid, being proportional to the wave amplitude squared, give rise to important contributions to M_z, i.e. B_{61}, B_{62}. At the same time a method for obtaining the complete wave drift damping matrix, and thereby for the first time the damping coefficients B_{16}, B_{26}, B_{66}, was given by Newman (1993). He applied a perturbation approach where the low-frequency oscillations of the floating body, which were assumed small, were superposed on the diffraction field. The theoretical framework was given without numerical examples.

In the present work a method is derived to evaluate the coefficients B_{i6}, $i = 1, 2, 6$, of the wave drift damping matrix, which is based on integral equations and with the motion referred to the relative coordinate system fixed to the body. In this frame of reference non-Newtonian forces are accounted for. A floating body performing a slow rotation about the vertical axis while being exposed to incoming monochromatic waves is considered. The rotation angle of the body may be finite, and may be an unspecified function of time. The angular velocity is assumed to be small compared to the wave frequency of the incoming waves, i.e. $\Omega/\omega \ll 1$, however. This justifies application of the method of multiple timescales. The fluid flow and the forces do then depend on the instantaneous wave angle.

The fluid is assumed to be homogenous and incompressible, and viscous effects are disregarded, such that potential theory may be applied. First an exact relation for the fluid pressure in the relative frame of reference is derived. The boundary value problem for the velocity potential is then developed. The method of multiple timescales, and perturbation expansion of the potential in terms of the wave amplitude and the slow angular velocity are then applied. Next integral equations are derived for the set of potentials, involving unknown quantities on the wetted body surface only. The integral equations are suitable for solution by means of a low-order panel method, which is applied here, giving efficient and robust numerical algoritms. The method is general and is applicable to bodies of arbitrary shape. To simplify the analysis, only linear diffraction effects are accounted for in this work. Thus, in the relative frame of reference the body is restrained. The water depth is assumed infinite, and the body is assumed to be wall-sided at the water line.

The time-averaged horizontal force and vertical moment may be obtained in different ways. The most usual procedures are either by integrating the pressure over the instantaneous wetted part of the body surface, or by applying conservation of linear and angular momentum. The latter method is applied here, giving as final result that F_x, F_y, M_z, B_{16}, B_{26}, and B_{66} are expressed in terms of the far-field amplitudes of the wave potentials and the dipole moments of the time-averaged second-order potential. The formulae are given in a form where all integrals are brought to a convergent form that is suitable for numerical evaluation.

A code for the complete method is developed, and numerical results are presented for two practical geometries, i.e. a vertical circular cylinder moving with its axis describing a circular path about the origin, and a ship. The calculated wave drift damping B_{66} is compared to viscous damping. In a realistic case of a ship of length 230 m and beam 41 m we find that the wave drift damping predominates when the wave amplitude is larger than 1.7 m. This result holds for all wave headings and for wave period less than 14 s (see §9.2). We also find that B_{16}, B_{26}, B_{66} may be one order of magnitude larger than F_{x0}, F_{y0}, M_{z0}. For the ship we obtain, for example, that $B_{66}/M_{z0} \simeq 200$ for a quartering sea with wavelength about half of the ship length. $(\Omega/\omega)B_{66}$ may then be about 25% of M_{z0} in a described practical case. (The details are explained in §9.3.)

We have not considered here the importance of wave drift damping compared to viscous damping (in the yaw mode) for an oil platform. However, it seems obvious that the relative effect of wave drift damping compared to viscous damping is approximately the same for slow rotation and slow translation for this geometry. Wave drift damping has proved to be significant for the latter mode of motion.

Our formulae for B_{i6} are in another form than those obtained by Newman. It is, however, possible by appropriate transformations to compare them. We find that our and his final formula for B_{66} are formally the same, except that the rotation angle of the body is finite in our analysis. For B_{i6}, $i = 1, 2$, however, there are discrepancies between our formula (7.11) and Newman's final result (5.6). The discrepancies are identified and discussed, see §7.

During the course of this work, the yaw problem was also being considered by Emmerhoff & Sclavounos (1993) and Emmerhoff (1994). They formulate the problem essentially in the absolute frame of reference, and present a solution for arrays of vertical cylinders. Both our and their methods, which were initiated independently, utilize the fact that the sum of the relative incident wave angle and the angle of rotation are independent of time. Preliminary results of our method were given by Grue & Palm (1994). In Sclavounos (1994) the equation governing the slow drift motion of a floating body, including the role of the wave drift damping matrix, is discussed.

The paper is organized as follows. In §2 the equation of motion for the fluid in the relative frame of reference is formulated, an exact relation for the pressure is derived, and the fast and the slow timescales are introduced. In §3 the boundary value problem is discussed and the perturbation potentials introduced. In §4 the resulting set of boundary value problems is solved by means of integral equations. In §5 the equations for the slowly rotating body are compared to those for the slow surge and sway problems. In §§6 and 7 expressions for the damping moment and the damping force, respectively, are derived. Section 8 is devoted to the balance of energy, and §9 describes numerical results. Finally, §10 contains concluding remarks.

2. The equation of motion and the pressure

We consider a floating body performing a slow time-dependent rotation about the vertical axis while being exposed to incoming waves. Two frames of references are introduced: one absolute frame of reference, $Ox^0y^0z^0$, fixed in space, and one relative frame of reference, $Oxyz$, following the slow rotation of the body. The vertical axis is defined by the $z = z^0$-axis, pointing upwards, with $z = z^0 = 0$ coinciding with the mean free surface. Unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are introduced along the x-, y-, z-axes, respectively. Let $\alpha(t)$ denote the rotation angle of the body relative to the x^0-axis, and $\boldsymbol{\Omega} = \Omega\boldsymbol{k} = \dot{\alpha}\boldsymbol{k}$ the angular velocity, where a dot denotes derivative with respect to time. $Ox^0y^0z^0$ and $Oxyz$ are then related by

$$x = x^0 \cos\alpha + y^0 \sin\alpha, \tag{2.1}$$

$$y = y^0 \cos\alpha - x^0 \sin\alpha, \tag{2.2}$$

$$z = z^0. \tag{2.3}$$

The fluid motion is considered in the relative frame of reference. Neglecting viscous effects the equation of motion for the fluid reads

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p - \rho g \nabla z + \boldsymbol{H} \tag{2.4}$$

where $\boldsymbol{v}$ and p denote the velocity and pressure of the fluid, respectively, ρ denotes the density, assumed constant, g denotes the acceleration due to gravity, and $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = -2\rho\boldsymbol{\Omega}\times\boldsymbol{v} - \rho\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{x} - \rho\dot{\boldsymbol{\Omega}}\times\boldsymbol{x}. \tag{2.5}$$

Here, the first term denotes the the Coriolis force, the second term the centrifugal force, and the third term the fictive force due to the angular acceleration.

A velocity field given by $-\boldsymbol{\Omega}\times\boldsymbol{x}$ is introduced when observing the fluid velocity from the relative frame of reference. Thus, $\boldsymbol{v}$ may be decomposed as $\boldsymbol{v} = \boldsymbol{v}' - \boldsymbol{\Omega}\times\boldsymbol{x}$. Assuming that $\boldsymbol{v}'$ is irrotational, this velocity may be obtained by the gradient of a velocity potential Φ', i.e. $\boldsymbol{v}' = \nabla\Phi'$. The equation of motion may then be written in the form

$$\nabla\left(\frac{\partial\Phi'}{\partial t} - \Omega\frac{\partial\Phi'}{\partial\theta} + \tfrac{1}{2}|\nabla\Phi'|^2\right) = \nabla\left(-\frac{p}{\rho} - gz\right) \tag{2.6}$$

where polar coordinates are introduced by $x = R\cos\theta$, $y = R\sin\theta$. By integration we obtain the following expression for the fluid pressure:

$$-\frac{p}{\rho} = \frac{\partial \Phi'}{\partial t} - \Omega\frac{\partial \Phi'}{\partial \theta} + \tfrac{1}{2}|\nabla\Phi'|^2 + gz + C(t) \tag{2.7}$$

where $C(t)$ is an arbitrary function of time. Both (2.6) and (2.7) are exact.

2.1. *Fast and slow timescales*

The wavenumber of the incoming waves, K, non-dimensionalized by the characteristic length of the floating body, l, is assumed to be of order unity. For deep-water waves this means that $\omega^2 l/g = O(1)$, where ω denotes the wave frequency in the absolute frame of reference. The rotation angle of the body may be of arbitrary magnitude, i.e. $\alpha(t) = O(1)$. The angular velocity Ω is, however, assumed to be much smaller than the wave frequency ω, i.e.

$$\Omega/\omega \ll 1. \tag{2.8}$$

Since the rotation angle is finite, this assumption implies that

$$\dot{\Omega}/\omega^2 = O(\Omega^2/\omega^2) \ll \Omega/\omega. \tag{2.9}$$

In the following analysis we shall apply a perturbation expansion in Ω/ω, retaining terms up to order Ω/ω. This means that the perturbed problem has two timescales: a fast timescale with characteristic time $1/\omega$, and a slow timescale with characteristic time $1/\Omega$. In obtaining for example the wave drift damping force and moment a time average over the fast timescale is applied.

3. The boundary value problems

Assuming that the fluid is incompressible, Φ' satisfies the Laplace equation in the fluid domain. It is convenient to decompose Φ' as

$$\Phi' = \phi_s + \Phi + \psi^{(2)} \tag{3.1}$$

where $\phi_s \equiv \Omega\chi_6$ denotes the potential generated by the body when there are no waves, Φ the linear wave potential proportional to the amplitude A of the incoming waves, and $\psi^{(2)}$ a time-averaged second-order potential proportional to the wave amplitude squared.

3.1. *Steady potentials* χ_1, χ_2, χ_6

The steady potential χ_6 appears in (3.1). Later, potentials χ_1 and χ_2 will also be required. The potentials χ_i, $i = 1,2,6$, satisfy the following boundary value problems:

$$\left.\begin{array}{lll} \nabla^2\chi_i = 0 & \text{in the fluid domain,} & \\ \dfrac{\partial\chi_i}{\partial z} = 0 & \text{at} \quad z = 0, & \\ \dfrac{\partial\chi_i}{\partial n} = n_i & \text{at} \quad S_B, & \\ |\nabla\chi_i| \rightarrow 0 & R \rightarrow \infty, & z \rightarrow -\infty, \end{array}\right\} \tag{3.2}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ denotes the unit normal at S_B, pointing out of the fluid, and $n_6 = \boldsymbol{n}\cdot(\boldsymbol{k}\times\boldsymbol{x}) = n_2 x - n_1 y$. The potentials χ_i, $i = 1,2,6$, are obtained by means of source distributions.

3.2. *The boundary conditions, Φ*

To obtain the free-surface condition for Φ, the individual time-derivative is applied to (2.7) at the free-surface elevation $z = \zeta$. After linearizing with respect to the wave amplitude we find

$$\frac{\partial^2 \Phi}{\partial t^2} - 2\Omega \frac{\partial^2 \Phi}{\partial \theta \partial t} + 2\nabla_h \phi_s \cdot \nabla_h \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \phi_s}{\partial z^2}\frac{\partial \Phi}{\partial t} + g\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{3.3}$$

where ∇_h denotes the horizontal gradient. Let us then introduce $\Phi = \mathrm{Re}[(Aig/\omega)\phi e^{i\omega t}]$. Noting that Φ and ϕ are functions of α, Ω, $\dot{\Omega}$, ..., we find for the partial time-derivative of Φ

$$\frac{\partial \Phi}{\partial t} = \mathrm{Re}\left\{\frac{Aig}{\omega}\left(i\omega\phi + \Omega\frac{\partial \phi}{\partial \alpha} + \dot{\Omega}\frac{\partial \phi}{\partial \Omega} + \ldots\right)e^{i\omega t}\right\}. \tag{3.4}$$

We now introduce the perturbation parameter $\epsilon \equiv \Omega/\omega$. By expanding (3.3) in terms of ϵ, retaining terms up to $O(\epsilon)$, the free-surface condition for ϕ reads

$$-K\phi + 2i\epsilon K\frac{\partial \phi}{\partial \alpha} - 2i\epsilon K\frac{\partial \phi}{\partial \theta} + 2i\epsilon K\nabla\phi \cdot \nabla_h \chi_6 + i\epsilon K\phi\nabla_h^2\chi_6 + \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0 \tag{3.5}$$

where $K = \omega^2/g$. In this paper we consider the diffraction problem, which means that the body has no motions in the relative frame of reference. The boundary condition for ϕ at the body is then given by

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{at} \quad S_B. \tag{3.6}$$

In addition, there is a radiation condition for $\phi - \phi_I$ as $R \to \infty$, where ϕ_I denotes the incoming wave potential defined below by (3.11), requiring that $\phi - \phi_I$ is composed of outgoing waves only. This is equivalent to requiring that there are no energy sources as $R \to \infty$, except the incoming waves.

3.3. *Perturbation procedure*

It is convenient to expand the potential ϕ in ϵ by

$$\phi = \phi^0 + \epsilon\phi^1 + \ldots; \tag{3.7}$$

ϕ^0 then satisfies

$$-K\phi^0 + \frac{\partial \phi^0}{\partial z} = 0 \quad \text{at} \quad z = 0, \tag{3.8}$$

$$\frac{\partial \phi^0}{\partial n} = 0 \quad \text{at} \quad S_B. \tag{3.9}$$

ϕ^0 is composed of the incoming wave potential, ϕ_I, and the scattering potential, ϕ_7, i.e. $\phi^0 = \phi_I + \phi_7$. In addition to the conditions at the free surface and the body boundary, ϕ_7 satisfies $|\nabla\phi_7| \to 0$ for $z \to -\infty$ and the radiation condition for $R \to \infty$, i.e.

$$\phi_7 = R^{-1/2}H^0(\theta)e^{Kz - iKR}(1 + O((KR)^{-1})) \tag{3.10}$$

where $H^0(\theta)$ denotes the far-field amplitude of the potential, which is determined by (4.19).

The incoming wave potential is given by, assuming water of infinite depth,

$$\phi_I = e^{Kz - iKR\cos(\beta - \theta)} \tag{3.11}$$

where β (which is a function of time) denotes the angle between the x-axis and the wave direction (in the relative frame of reference). β is related to the angle between the x^0-direction and the wave direction, β^0, (in the absolute frame of reference) by $\beta = \beta^0 - \alpha$. It is obvious that ϕ^0 is a function of $\beta = \beta^0 - \alpha$. This means that $\partial\phi^0/\partial\alpha = -\partial\phi^0/\partial\beta$. The free-surface boundary condition for ϕ^1 then becomes

$$-K\phi^1 + \frac{\partial\phi^1}{\partial z} = 2\mathrm{i}K\frac{\partial\phi^0}{\partial\beta} + 2\mathrm{i}K\frac{\partial\phi^0}{\partial\theta} - 2\mathrm{i}K\nabla_h\phi^0\cdot\nabla_h\chi_6 - \mathrm{i}K\phi^0\nabla_h^2\chi_6 \quad \text{at} \quad z = 0. \tag{3.12}$$

In the diffraction problem the body boundary condition is

$$\frac{\partial\phi^1}{\partial n} = 0 \quad \text{at} \quad S_B. \tag{3.13}$$

In addition, ϕ^1 satisfies the condition of outgoing waves as $R \to \infty$, and $|\nabla\phi^1| \to 0$ for $z \to -\infty$.

It is convenient to decompose the boundary value problem for ϕ^1 by introducing

$$\phi^1 = \phi^{11} + \phi^{12} + \phi^{13} \tag{3.14}$$

where ϕ^{11}, ϕ^{12}, ϕ^{13} satisfy the following set of boundary value problems:

$$-K\phi^{11} + \frac{\partial\phi^{11}}{\partial z} = 2\mathrm{i}K\frac{\partial\phi^0}{\partial\beta} \quad \text{at} \quad z = 0, \tag{3.15}$$

$$\frac{\partial\phi^{11}}{\partial n} = 0 \quad \text{at} \quad S_B, \tag{3.16}$$

$$-K\phi^{12} + \frac{\partial\phi^{12}}{\partial z} = 2\mathrm{i}K\frac{\partial\phi^0}{\partial\theta} \quad \text{at} \quad z = 0, \tag{3.17}$$

$$-K\phi^{13} + \frac{\partial\phi^{13}}{\partial z} = -2\mathrm{i}K\nabla_h\phi^0\cdot\nabla_h\chi_6 - \mathrm{i}K\phi^0\nabla_h^2\chi_6 \quad \text{at} \quad z = 0, \tag{3.18}$$

$$\frac{\partial\phi^{13}}{\partial n} = -\frac{\partial\phi^{12}}{\partial n} \quad \text{at} \quad S_B; \tag{3.19}$$

ϕ^{13} satisfies, in addition to (3.18)–(3.19), $|\nabla\phi^{13}| \to 0$ for $z \to -\infty$, and the radiation condition for $R \to \infty$, i.e.

$$\phi^{13} = R^{-1/2}H^{13}(\theta)\mathrm{e}^{Kz-\mathrm{i}KR}(1 + O((KR)^{-1})) \tag{3.20}$$

where $H^{13}(\theta)$ denotes the far-field amplitude of the potential, which is determined by (4.21).

The potentials ϕ^{11} and ϕ^{12} may be given in the form (Nossen *et al.* 1991, equation (34); Emmerhoff & Sclavounos 1992, equation (38))

$$\phi^{11} = 2\mathrm{i}K\frac{\partial^2\phi^0}{\partial K\,\partial\beta}, \tag{3.21}$$

$$\phi^{12} = 2\mathrm{i}K\frac{\partial^2\phi^0}{\partial K\,\partial\theta}. \tag{3.22}$$

The solutions (3.21) and (3.22) contain secular terms. This means that the potentials ϕ^{11} and ϕ^{12} become infinitely large for $R \to \infty$. This does not, however, lead to any mathematical problem as long as the value of R is large, but finite. Nevertheless, we want to utilize the far-field form of the potentials involved to deduce the final form of the integral equation and convenient formulae for the damping coefficients.

Therefore, in our derivations all relevant integrals are brought to convergent forms before $R \to \infty$.

We further note that $\phi^{11} + \phi^{12} = 2\mathrm{i}K\partial/\partial K(\partial\phi_7/\partial\beta + \partial\phi_7/\partial\theta)$ represent outgoing waves for large R.

3.4. *The boundary conditions, $\psi^{(2)}$*

The second-order potential $\psi^{(2)}$ appears in the formulae for the second-order fluid pressure and for the mean force and moment, always multiplied by the slow angular velocity Ω. To leading order in Ω it is then sufficient to consider the boundary value problem for $\psi^{(2)}$ when $\Omega = 0$. The free surface condition for $\psi^{(2)}$ then reads

$$\frac{\partial\psi^{(2)}}{\partial z} = -\overline{\frac{1}{g}\frac{\partial}{\partial t}\nabla\Phi\cdot\nabla\Phi} + \overline{\frac{1}{g^2}\frac{\partial\Phi}{\partial t}\frac{\partial^3\Phi}{\partial z\partial t^2}} + \overline{\frac{1}{g}\frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial z^2}} = -\frac{A^2 g}{2\omega}\mathrm{Im}\left(\phi^0\frac{\partial^2\phi^{0*}}{\partial z^2}\right) \quad \text{at } z = 0 \tag{3.23}$$

where a bar denotes time-average and an asterisk complex conjungate. In the diffraction problem, $\partial\psi^{(2)}/\partial n = 0$ at the body surface. In addition, $|\nabla\psi^{(2)}| \to 0$ for $R \to \infty$, or $z \to -\infty$.

The solution for $\psi^{(2)}$ may be obtained by an integral equation. The analysis below shows, however, that only the boundary conditions for $\psi^{(2)}$ are required to find the mean force and moment acting on the floating body. Thus, the complete solution for $\psi^{(2)}$ is not needed. In the general case $\psi^{(2)}$ also satisfies a non-trivial boundary condition at the body boundary. A complete discussion of the significance of $\psi^{(2)}$, and how to obtain the potential, is given by Grue & Palm (1993).

4. Integral equations

As will be shown in §§6 and 7, the far-field amplitudes H^0 (see (3.10) and (4.19)), including derivatives of H^0 with respect to β, θ, and K in combinations up to second order, and H^{13} (see (3.20) and (4.21)) are required to find the damping coefficients B_{i6}. Furthermore, χ_i $(i = 1, 2)$ and $\mathrm{Im}(\phi^0\phi^{0*}_{zz})$ must be determined in the formula for B_{16}, B_{26}. While H^0 is determined by ϕ^0, H^{13} is most conveniently determined by the potentials χ_6 (see §3.1), ϕ^0, and $\phi^1 - \phi^{11} = \phi^{12} + \phi^{13}$. We shall in what follows deduce integral equations for the two latter potentials.

4.1. *The potential ϕ^0*

To solve the boundary value problem for ϕ^0 we first introduce a Green function, $G^0(a, b, c, x, y, z)$, being a sink at $\boldsymbol{x} = \boldsymbol{a} = (a, b, c)$, satisfying the free-surface boundary condition (3.8). This Green function may be written by, see e.g. Wehausen & Laitone (1960, equation 13.17),

$$G^0 = \frac{1}{r} + \int_0^\infty \frac{k + K}{k - K}\mathrm{e}^{k(z+c)}J_0(kR')\mathrm{d}k, \tag{4.1}$$

where $r = |\boldsymbol{x} - \boldsymbol{a}|$, $R' = [(x - a)^2 + (y - b)^2]^{1/2}$, J_0 denotes the Bessel function of first kind and order zero, and the path of integration is above the pole at $k = K$. For $R \to \infty$, G^0 takes the form

$$G^0 = R^{-1/2}h^0(\theta)\mathrm{e}^{Kz - \mathrm{i}KR}(1 + O((KR^{-1}))) \tag{4.2}$$

where

$$h^0 = (8\pi K)^{1/2}\exp[K(c + \mathrm{i}a\cos\theta + \mathrm{i}b\sin\theta) - \mathrm{i}\pi/4]. \tag{4.3}$$

By applying Green's theorem to ϕ^0 and G^0 it may be shown that ϕ^0 satisfies, see e.g. Nossen *et al.* (1991, equation 40),

$$\int_{S_B} \phi^0 \frac{\partial G^0}{\partial n} \mathrm{d}S - 4\pi\phi_I = \begin{cases} -2\pi\phi^0(\boldsymbol{x}), & \boldsymbol{x} \in S_B \\ -4\pi\phi^0(\boldsymbol{x}), & \boldsymbol{x} \in \mathscr{V}, \end{cases} \tag{4.4}$$

where $\mathscr{V}$ denotes the fluid volume enclosed by the body surface, S_B, the free surface, S_F, and the vertical circular cylinder, $S(R)$, with radius R. The integration is over the (a, b, c)-variables. The first case is an integral equation for ϕ^0.

Proper forms of the derivatives of the potential ϕ^0 for numerical use are obtained by means of integral equations. For example, $\partial\phi^0/\partial\beta$ may be determined by differentiating the integral equation (4.4) with respect to β. An integral equation for $\partial^2\phi^0/\partial K \partial\beta$ is obtained by differentiating (4.4) for ϕ^0 with respect to β and K. The result is

$$\int_{S_B} \phi^0_{\beta K} \frac{\partial G^0}{\partial n} \mathrm{d}S + \int_{S_B} \phi^0_\beta \frac{\partial^2 G^0}{\partial n \partial K} \mathrm{d}S - 4\pi\phi_{I,\beta K} = \begin{cases} -2\pi\phi^0_{\beta K}(\boldsymbol{x}), & \boldsymbol{x} \in S_B \\ -4\pi\phi^0_{\beta K}(\boldsymbol{x}), & \boldsymbol{x} \in \mathscr{V}, \end{cases} \tag{4.5}$$

where $\phi^0_\beta \equiv \partial\phi^0/\partial\beta$, $\phi^0_{\beta K} \equiv \partial^2\phi^0/\partial K \partial\beta$, $\phi_{I,\beta K} \equiv \partial^2\phi_I/\partial K \partial\beta$. This equation determines $\phi^0_{\beta K}$.

A formula for $\phi^0_{\theta K} = \partial^2\phi^0/\partial K \partial\theta$ may next be obtained by differentiating (4.4) with respect to θ and K, giving

$$\int_{S_B} \phi^0_K \frac{\partial^2 G^0}{\partial\theta\partial n} \mathrm{d}S + \int_{S_B} \phi^0 \frac{\partial^3 G^0}{\partial K \partial\theta\partial n} \mathrm{d}S - 4\pi\phi_{I,\theta K} = -4\pi\phi^0_{\theta K}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathscr{V}. \tag{4.6}$$

(Note that $\partial/\partial n = \boldsymbol{n}(\boldsymbol{a}) \cdot \nabla \boldsymbol{a}$, while $\partial/\partial\theta = x\partial/\partial y - y\partial/\partial x$.)

4.2. *The potential* $\phi^1 - \phi^{11}$

To find an integral equation for $\phi^1 - \phi^{11}$ we first introduce an auxillary function, $G^1(a, b, c, x, y, z)$, regular for $c < 0$, $z < 0$, satisfying the following free-surface boundary condition:

$$-KG^1 + \frac{\partial G^1}{\partial c} = 2\mathrm{i}K \frac{\partial G^0}{\partial\tilde{\theta}} \quad \text{at} \quad c = 0 \tag{4.7}$$

where $\tilde{\theta}$ is defined by $a = \tilde{R}\cos\tilde{\theta}$, $b = \tilde{R}\sin\tilde{\theta}$, $\tilde{R}^2 = a^2 + b^2$. It may be shown that G^1 may be expressed in terms of G^0 by

$$G^1 = 2\mathrm{i}K \frac{\partial^2 G^0}{\partial\tilde{\theta}\partial K}. \tag{4.8}$$

We first apply Green's theorem to $\psi = \phi^1 - \phi^{11} = \phi^{12} + \phi^{13}$ and G^0, giving

$$\int_{S_B} \psi \frac{\partial G^0}{\partial n} \mathrm{d}S + \int_{S_F + S(R)} \left(\psi \frac{\partial G^0}{\partial n} - G^0 \frac{\partial\psi}{\partial n} \right) \mathrm{d}S = \begin{cases} -2\pi\psi(\boldsymbol{x}), & \boldsymbol{x} \in S_B \\ -4\pi\psi(\boldsymbol{x}), & \boldsymbol{x} \in \mathscr{V}. \end{cases} \tag{4.9}$$

By then applying the free-surface boundary conditions for ϕ^{12}, ϕ^{13} and G^0, and the body boundary condition for ψ, (4.9) reduces to

$$\int_{S_B} \psi \frac{\partial G^0}{\partial n} \mathrm{d}S + 2\mathrm{i}K \int_{S_F} G^0 \left(-\frac{\partial\phi^0}{\partial\tilde{\theta}} + \nabla_h\phi^0 \cdot \nabla_h\chi_6 + \tfrac{1}{2}\phi^0\nabla_h^2\chi_6 \right) \mathrm{d}S$$
$$+ \int_{S(R)} \left(\psi \frac{\partial G^0}{\partial n} - G^0 \frac{\partial\psi}{\partial n} \right) \mathrm{d}S = \begin{cases} -2\pi\psi(\boldsymbol{x}), & \boldsymbol{x} \in S_B \\ -4\pi\psi(\boldsymbol{x}), & \boldsymbol{x} \in \mathscr{V}. \end{cases} \tag{4.10}$$

The integrals over S_F and $S(R)$ are not in a final form, and may be further developed.

To do this we apply Green's theorem to ϕ^0 and G^1. By introducing the boundary conditions for ϕ^0 and G^1 at the free surface we obtain

$$\int_{S_B} \phi^0 \frac{\partial G^1}{\partial n} \mathrm{d}S + 2\mathrm{i}K \int_{S_F} \phi^0 \frac{\partial G^0}{\partial \tilde{\theta}} \mathrm{d}S + \int_{S(R)} \left(\phi^0 \frac{\partial G^1}{\partial n} - G^1 \frac{\partial \phi^0}{\partial n} \right) \mathrm{d}S = 0. \tag{4.11}$$

Subtracting (4.11) from (4.10) gives

$$\begin{aligned} &\int_{S_B} \left(\psi \frac{\partial G^0}{\partial n} - \phi^0 \frac{\partial G^1}{\partial n} \right) \mathrm{d}S + \int_{S(R)} \left(\psi \frac{\partial G^0}{\partial n} - G^0 \frac{\partial \psi}{\partial n} \right) \mathrm{d}S \\ &+2\mathrm{i}K \int_{S_F} \left[-\frac{\partial}{\partial \tilde{\theta}} (G^0 \phi^0) + G^0 (\nabla_h \phi^0 \cdot \nabla_h \chi_6 + \tfrac{1}{2} \phi^0 \nabla_h^2 \chi_6) \right] \mathrm{d}S \\ &+I = \begin{cases} -2\pi \psi(\boldsymbol{x}), & \boldsymbol{x} \in S_B \\ -4\pi \psi(\boldsymbol{x}), & \boldsymbol{x} \in \mathscr{V}, \end{cases} \end{aligned} \tag{4.12}$$

where

$$I = \int_{S(R)} \left(\phi^{12} \frac{\partial G^0}{\partial n} - G^0 \frac{\partial \phi^{12}}{\partial n} - \phi^0 \frac{\partial G^1}{\partial n} + G^1 \frac{\partial \phi^0}{\partial n} \right) \mathrm{d}S. \tag{4.13}$$

It is shown in Appendix A that

$$I = 8\pi \mathrm{i}K \frac{\partial^2 \phi_I}{\partial \beta \partial K}. \tag{4.14}$$

The free-surface integral in (4.12) may be further developed by noting that

$$\begin{aligned} &-\frac{\partial}{\partial \tilde{\theta}} (G^0 \phi^0) + G^0 (\nabla_h \phi^0 \cdot \nabla_h \chi_6 + \tfrac{1}{2} \phi^0 \nabla_h^2 \chi_6) \\ &\quad = \nabla_h \cdot [G^0 \phi^0 (\nabla_h \chi_6 - \boldsymbol{k} \times \boldsymbol{x})] - \phi^0 (\nabla_h G^0 \cdot \nabla_h \chi_6 + \tfrac{1}{2} G^0 \nabla_h^2 \chi_6). \end{aligned} \tag{4.15}$$

By application of Gauss' theorem, assuming that the body is wall-sided at the water line, we then obtain

$$\begin{aligned} &\int_{S_F} \left[-\frac{\partial}{\partial \tilde{\theta}} (G^0 \phi^0) + G^0 (\nabla_h \phi^0 \cdot \nabla_h \chi_6 + \tfrac{1}{2} \phi^0 \nabla_h^2 \chi_6) \right] \mathrm{d}S \\ &\quad = -\int_{S_F} \phi^0 (\nabla_h G^0 \cdot \nabla_h \chi_6 + \tfrac{1}{2} G^0 \nabla_h^2 \chi_6) \mathrm{d}S + \int_{C_B + C(R)} \phi^0 G^0 \left[\frac{\partial \chi_6}{\partial n} - \boldsymbol{n} \cdot (\boldsymbol{k} \times \boldsymbol{x}) \right] \mathrm{d}l \end{aligned} \tag{4.16}$$

where C_B and $C(R)$ denote the contours at $z = 0$ of S_B and $S(R)$, respectively. The integral along C_B vanishes due to the boundary condition (3.2) for χ_6 at S_B. The integrand at $C(R)$ disappears as $R \to \infty$. The integral over S_F is now in a form which converges very rapidly. Furthermore, $\phi^{13} \partial G^0 / \partial n - G^0 \partial \phi^{13} / \partial n$ at $S(R)$ vanishes as $R \to \infty$ since ϕ^{13} and G^0 satisfy the same radiation condition. By then combining (4.12), (4.14), (4.16), we arrive at the final result for $\phi^1 - \phi^{11}$:

$$\begin{aligned} &\int_{S_B} (\phi^1 - \phi^{11}) \frac{\partial G^0}{\partial n} \mathrm{d}S + 8\pi \mathrm{i}K \frac{\partial^2 \phi_I}{\partial \beta \partial K} - \int_{S_B} \phi^0 \frac{\partial G^1}{\partial n} \mathrm{d}S \\ &-2\mathrm{i}K \int_{S_F} \phi^0 (\nabla_h G^0 \cdot \nabla_h \chi_6 + \tfrac{1}{2} G^0 \nabla_h^2 \chi_6) \mathrm{d}S = \begin{cases} -2\pi (\phi^1 - \phi^{11})(\boldsymbol{x}), & \boldsymbol{x} \in S_B \\ -4\pi (\phi^1 - \phi^{11})(\boldsymbol{x}), & \boldsymbol{x} \in \mathscr{V}. \end{cases} \end{aligned} \tag{4.17}$$

In the first case (4.17) is an integral equation for $\phi^1 - \phi^{11}$.

The damping coefficients B_{i6} may now be obtained from the potential ϕ^0, its derivatives, and $\phi^1 - \phi^{11}$. If, in addition, for example, the linear exciting force on the body is to be determined, the complete potential ϕ^1 is required. This potential may be obtained by adding (4.5) multiplied by 2iK and (4.17) with the result

$$\int_{S_B} \phi^1 \frac{\partial G^0}{\partial n} \mathrm{d}S - \int_{S_B} \left(\phi^0 \frac{\partial G^1}{\partial n} + 2\mathrm{i}K \frac{\partial \phi^0}{\partial \beta} \frac{\partial^2 G^0}{\partial n \partial K} \right) \mathrm{d}S$$
$$-2\mathrm{i}K \int_{S_F} \phi^0 (\nabla_h G^0 \cdot \nabla_h \chi_6 + \tfrac{1}{2} G^0 \nabla_h^2 \chi_6) \mathrm{d}S = \begin{cases} -2\pi\phi^1(\boldsymbol{x}), & \boldsymbol{x} \in S_B \\ -4\pi\phi^1(\boldsymbol{x}), & \boldsymbol{x} \in \mathscr{V}. \end{cases} \tag{4.18}$$

In the first case (4.18) is an integral equation for ϕ^1.

4.3. *The far-field amplitudes H^0 and H^{13}*

Consider now far-field amplitudes H^0 and H^{13}. By introducing the far-field form of G^0 into (4.4), we obtain for H^0

$$H^0 = -\frac{1}{4\pi} \int_{S_B} \phi^0 \frac{\partial h^0}{\partial n} \mathrm{d}S \tag{4.19}$$

where h^0 is given by (4.3). Derivatives of H^0 with respect to β, θ, K, or combinations of these variables, are easily obtained. H^0_β is for example found from $H^0_\beta = -1/(4\pi) \int_{S_B} \phi^0_\beta h^0_n \mathrm{d}S$. Consider now ϕ^{13}, which is obtained by subtracting ϕ^{12}, obtained by (4.6), from $\phi^1 - \phi^{11}$. This gives

$$4\pi\phi^{13}(\boldsymbol{x}) = \int_{S_B} (\phi^{11} - \phi^1) \frac{\partial G^0}{\partial n} \mathrm{d}S + 2\mathrm{i}K \int_{S_B} \phi^0_K \frac{\partial^2 G^0}{\partial n \partial \theta} \mathrm{d}S$$
$$+ 2\mathrm{i}K \int_{S_F} \phi^0 (\nabla_h \chi_6 \cdot \nabla_h G^0 + \tfrac{1}{2} G^0 \nabla_h^2 \chi_6) \mathrm{d}S, \qquad \boldsymbol{x} \in \mathscr{V}. \tag{4.20}$$

By then introducing the far-field form of G^0 we obtain

$$H^{13} = -\frac{1}{4\pi} \int_{S_B} (\phi^1 - \phi^{11}) \frac{\partial h^0}{\partial n} \mathrm{d}S + \frac{\mathrm{i}K}{2\pi} \int_{S_B} \phi^0_K \frac{\partial}{\partial n} \frac{\partial h^0}{\partial \theta} \mathrm{d}S$$
$$+ \frac{\mathrm{i}K}{2\pi} \int_{S_F} \phi^0 (\nabla_h \chi_6 \cdot \nabla_h h^0 + \tfrac{1}{2} h^0 \nabla_h^2 \chi_6) \mathrm{d}S \tag{4.21}$$

with h^0 given by (4.3).

4.4. *Remarks on the numerical procedure*

The set of potentials and source distributions is solved by means of their respective integral equations, applying a low-order panel method as the numerical method. The body surface and the free surface are discretized by quadrilaterals, and the potential or source strength is taken as constant at each panel. The Green function G^0 and its derivatives involved in the integral equations have singularities $\nabla(1/r)$, $\nabla(1/r')$, $1/r$, $1/r'$, where $r = |\boldsymbol{x} - \boldsymbol{a}|$ and $r' = [(x-a)^2 + (y-b)^2 + (z+c)^2]^{1/2}$. These singularities are integrated separately over each panel by analytical methods, see Newman (1985). Otherwise a midpoint rule is applied for numerical integration.

The quantity $\nabla_h^2 \chi_6$ at $z = 0$ appears in the integral equation for $\phi^1 - \phi^{11}$. It follows from the Laplace equation that we may instead evaluate $-\partial^2 \chi_6 / \partial z^2$, which is here obtained by numerical differentiation of $-\partial \chi_6 / \partial z$, where we utilize that $\partial \chi_6 / \partial z = 0$ at $z = 0$, giving quite robust predictions of the quantity.

The free-surface integrals

$$\int_{S_F} (\chi_i - x_i)\mathrm{Im}(\phi^0\phi^{0*}_{zz})\mathrm{d}S, \qquad i = 1, 2, \tag{4.22}$$

in the formula (7.12) for B_{16}, B_{26} are best obtained when ϕ_7 is represented by a source distribution σ_7 over the body surface, i.e. $\phi_7 = \int_{S_7} \sigma_7 G^0 \mathrm{d}S$. We then find $\phi_{7,zz} = \int_{S_B} \sigma_7 G^0_{zz} \mathrm{d}S$. When integrating G_{zz} over quadrilaterals, the singular terms are treated separately by analytic formulae. The integrals of $(\partial/\partial z)[(\partial/\partial z)(1/r + 1/r')]_{z=0}$ are obtained by using numerical difference when the source point is close to the quadrilateral and by four-points Gauss integration otherwise. This method is found to work well to predict (4.22).

We further note that (4.22) denote the horizontal dipole moments of the potential $\psi^{(2)}$ for $R \to \infty$, see Grue & Palm (1993, equations 77–79). These integrals have relatively quick convergence, since $\mathrm{Im}(\phi^0\phi^{0*}_{zz})$ quite rapidly tends to zero with increasing distance from the body. The integrals (4.22) may be transformed to integrals containing first derivatives of ϕ^0, where the latter are more robust quantities to evaluate than second derivatives close to the body surface when using the low-order panel method. Such a transformation leads, however, to two integrals over S_F and along $C(R)$, respectively, which are unbounded as the truncation radius increases, and accurate evaluation of their sum may not be trivial.

5. Comparison with the translatory case

At this point it is worthwhile to point out the close connection between the problems with slow rotation and slow translation. The equations for the slow surge and sway problems may be deduced from the more general yaw problem. This also gives a possibility for checking the equations derived above. We shall therefore derive the equation for surge. A similar procedure may also be used for the sway problem.

To obtain the equation for surge we situate the body, with finite dimensions, at $y \to -\infty$ (and $x = 0$). The rotation of the body with respect to the origin then corresponds to a translation along the x-direction, such that $-\Omega y \to U$, where U is a small velocity along the positive x-direction. Then

$$\Omega\chi_6 \to U\chi_1, \quad \Omega\frac{\partial}{\partial\theta} \to U\frac{\partial}{\partial x}, \quad \Omega\frac{\partial\phi^0}{\partial\beta} \to \mathrm{i}KU\cos\beta\phi^0, \quad \Omega G^1 \to -2\mathrm{i}KU\frac{\partial^2 G^0}{\partial x\partial K}. \tag{5.1}$$

The fluid velocity is then given by $\boldsymbol{v} = -U\boldsymbol{i} + \nabla\Phi'$, where

$$\Phi' = \mathrm{Re}[(A\mathrm{i}g/\omega)\phi\mathrm{e}^{\mathrm{i}\omega t}] + U\chi_1 + \psi^{(2)}. \tag{5.2}$$

By introducing $\tau(U) = \omega U/g$, the potential ϕ is expanded as

$$\phi = \phi^0 + \tau(U)\phi^{1U} + \dots. \tag{5.3}$$

The potential ϕ^0 is determined by (4.4), and χ_1 and $\psi^{(2)}$ are defined by the boundary value problems formulated in §§3.1 and 3.4, respectively. To obtain the potential ϕ^{1U} we first relate ϕ^{1U} to ϕ^1 by

$$\frac{\Omega}{\omega}\phi^1 = \frac{KU}{\omega}\phi^{1U} = \tau(U)\phi^{1U}. \tag{5.4}$$

The boundary condition for ϕ^{1U} at the free surface may then be obtained by introducing (5.1) and (5.4) in (3.12), giving

$$-K\phi^{1U}+\frac{\partial\phi^{1U}}{\partial z}=-2K\cos\beta\phi^0+2\mathrm{i}\frac{\partial\phi^0}{\partial x}-2\mathrm{i}\nabla_h\phi^0\cdot\nabla_h\chi_1-\mathrm{i}\phi^0\nabla_h^2\chi_1 \quad \text{at} \quad z=0. \tag{5.5}$$

In addition, ϕ^{1U} satisfies $\partial\phi^{1U}/\partial n=0$ at the body, $\nabla\phi^{1U}\to 0$ for $z\to-\infty$, and a radiation condition for $R\to\infty$. The boundary value problem for ϕ^{1U} may be solved by means of an integral equation, which is obtained by introducing (5.1), (5.4) and (5.5) into (4.18), giving

$$\begin{aligned}&\int_{S_B}\phi^{1U}\frac{\partial G^0}{\partial n}\mathrm{d}S-\int_{S_B}\phi^0\left(2K\cos\beta\frac{\partial^2G^0}{\partial n\partial K}-2\mathrm{i}\frac{\partial}{\partial n}\frac{\partial^2G^0}{\partial x\partial K}\right)\mathrm{d}S\\&-2\mathrm{i}\int_{S_F}\phi^0(\nabla_hG^0\cdot\nabla_h\chi_1+\tfrac{1}{2}G^0\nabla_h^2\chi_1)\mathrm{d}S=\begin{cases}-2\pi\phi^{1U}(\boldsymbol{x}), & \boldsymbol{x}\in S_B\\ -4\pi\phi^{1U}(\boldsymbol{x}), & \boldsymbol{x}\in\mathscr{V},\end{cases}\end{aligned} \tag{5.6}$$

where the first case is an integral equation for ϕ^{1U}. Wave radiation and wave diffraction due to a body with a small forward speed were studied by Nossen *et al.* (1991), who applied the decomposition $\phi=\phi^0(\nu)+\tau(U)\tilde{\phi}^1(\nu)$ of the potential ϕ, where $\nu=K-2K\cos\beta\tau(U)$. They arrived at the following integral equation for $\tilde{\phi}^1(\nu)$:

$$\begin{aligned}&\int_{S_B}\tilde{\phi}^1\frac{\partial G^0}{\partial n}\mathrm{d}S-\int_{S_B}2\mathrm{i}\phi^0\frac{\partial}{\partial n}\frac{\partial^2G^0}{\partial x\partial\nu}\mathrm{d}S\\&-2\mathrm{i}\int_{S_F}\phi^0(\nabla_hG^0\cdot\nabla_h\chi_1+\tfrac{1}{2}G^0\nabla_h^2\chi_1)\mathrm{d}S=\begin{cases}-2\pi\tilde{\phi}^1(\boldsymbol{x}), & \boldsymbol{x}\in S_B\\ -4\pi\tilde{\phi}^1(\boldsymbol{x}), & \boldsymbol{x}\in\mathscr{V}.\end{cases}\end{aligned} \tag{5.7}$$

Now,

$$\phi=\phi^0(\nu)+\tau(U)\tilde{\phi}^1(\nu)=\phi^0(K)+\tau(U)[\tilde{\phi}^1(K)-2K\cos\beta\phi^0_\nu(K)]+O(\tau(U)^2). \tag{5.8}$$

The potential $-2K\cos\beta\phi^0_\nu(K)$ may be obtained by the following integral equation:

$$\int_{S_B}(-2K\cos\beta\phi^0_K)\frac{\partial G^0}{\partial n}\mathrm{d}S-\int_{S_B}2K\cos\beta\phi^0\frac{\partial^2G^0}{\partial n\partial K}\mathrm{d}S=\begin{cases}-2\pi\phi^{1U}(\boldsymbol{x}), & \boldsymbol{x}\in S_B\\ -4\pi\phi^{1U}(\boldsymbol{x}), & \boldsymbol{x}\in\mathscr{V}.\end{cases} \tag{5.9}$$

By adding (5.7), with ν replaced by K, and (5.9) we obtain the integral equation (5.6) for $\phi^{1U}=\tilde{\phi}^1(K)-2K\cos\beta\phi^0_\nu(K)$, as expected.

6. The damping moment

It is of principal interest to derive expressions for the wave drift damping force and moment. These quantities may be obtained in different ways. The most usual procedures are either by pressure integration over the wetted body surface, or by applying conservation of linear and angular momentum. Both methods have their advantages. Here we shall apply conservation of linear and angular momentum, resulting in compact formulae which are easy to evaluate numerically by a low-order boundary element method.

First we consider the time-averaged moment about the vertical axis, M_z. Conservation of angular momentum gives the following relation:

$$\begin{aligned}M_z&\equiv\boldsymbol{k}\cdot\overline{\int_{S_B}p(\boldsymbol{x}\times\boldsymbol{n})\mathrm{d}S}\\&=\boldsymbol{k}\cdot\overline{\left[-\rho\frac{\mathrm{d}}{\mathrm{d}t}\int_V\boldsymbol{x}\times\boldsymbol{v}'\,\mathrm{d}V-\int_{S(R)}p\boldsymbol{x}\times\boldsymbol{n}\,\mathrm{d}S-\rho\int_{S(R)}\boldsymbol{x}\times\boldsymbol{v}'\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S\right]}.\end{aligned} \tag{6.1}$$

As control surface a vertical circular cylinder $S(R)$ with axis passing through the origin is applied, where we have that $\boldsymbol{k}\cdot(\boldsymbol{x}\times\boldsymbol{n})=0$ and $\boldsymbol{v}\cdot\boldsymbol{n}=\boldsymbol{v}'\cdot\boldsymbol{n}$. Introducing $\boldsymbol{v}'=\nabla\Phi'$ we obtain

$$M_z=\rho\Omega\frac{\partial}{\partial\beta}\overline{\int_V \boldsymbol{k}\cdot(\boldsymbol{x}\times\nabla\Phi')\mathrm{d}V}-\rho\int_{S(R)}\overline{\Phi'_\theta\Phi'_n}\mathrm{d}S \tag{6.2}$$

where $\Phi'_\theta\equiv\partial\Phi'/\partial\theta$, $\Phi'_n\equiv\partial\Phi'/\partial n$. In obtaining M_z, terms proportional to Ω^2, A^3 are disregarded. The integral in (6.2) over the fluid volume may be rewritten by applying Gauss' theorem, i.e.

$$\overline{\boldsymbol{k}\cdot\int_V \boldsymbol{x}\times\nabla\Phi'\mathrm{d}V}=\overline{\int_{S_B+S_F}\boldsymbol{k}\cdot(\boldsymbol{x}\times\boldsymbol{n})\Phi'\mathrm{d}S}=\int_{S_B}n_6\psi^{(2)}\mathrm{d}S-\int_{S_F}\overline{\zeta_\theta\Phi'}\mathrm{d}S. \tag{6.3}$$

Here we have applied $\boldsymbol{k}\cdot(\boldsymbol{x}\times\boldsymbol{n})=n_6$ at S_B, $\boldsymbol{k}\cdot(\boldsymbol{x}\times\boldsymbol{n})=-\zeta_\theta$ at S_F, $\boldsymbol{k}\cdot(\boldsymbol{x}\times\boldsymbol{n})=0$ at $S(R)$, and that $\overline{\Phi'}=\psi^{(2)}$. By applying Green's theorem to χ_6 and $\psi^{(2)}$ and exploiting the boundary conditions for χ_6 and $\psi^{(2)}$, we may show that

$$\int_{S_B}n_6\psi^{(2)}\mathrm{d}S=\int_{S_F+S_B}\chi_6\psi_n^{(2)}\mathrm{d}S=-\frac{A^2g}{2\omega}\int_{S_F}\chi_6\mathrm{Im}(\phi^0\phi_{zz}^{0*})\mathrm{d}S \tag{6.4}$$

where we have applied (3.23) and $\psi_n^{(2)}=0$ at S_B. We then expand the yaw moment in ϵ, i.e. $M_z=M_{z0}-\epsilon B_{66}$, where a minus sign is adopted since $-\epsilon B_{66}$ appears as a damping moment. Introducing $\phi=\phi^0+\epsilon\phi^1$ we obtain the following expression for M_{z0}:

$$\frac{M_{z0}}{\rho g A^2}=-\frac{1}{2K}\int_{S(R)}\mathrm{Re}\left[\phi_\theta^0\phi_n^{0*}\right]\mathrm{d}S. \tag{6.5}$$

This is a well-known result which agrees with Newman (1967). For B_{66} we find

$$\frac{B_{66}}{\rho g A^2}=\frac{1}{2}\frac{\partial}{\partial\beta}\int_{S_F}\left[\chi_6\mathrm{Im}(\phi^0\phi_{zz}^{0*})+\mathrm{Im}(\phi_\theta^0\phi^{0*})\right]\mathrm{d}S+\frac{1}{2K}\int_{S(R)}\mathrm{Re}\left[\phi_\theta^0\phi_n^{1*}+\phi_\theta^1\phi_n^{0*}\right]\mathrm{d}S. \tag{6.6}$$

Since the individual integrals in (6.6) do not converge for $R\to\infty$ (but their sum converges), it is appropriate to rewrite the integral over S_F in (6.6). By applying Gauss' theorem twice, and exploiting the boundary conditions for ϕ^0, ϕ^1, and χ_6 at S_F and S_B, we may show that

$$\frac{1}{2}\frac{\partial}{\partial\beta}\int_{S_F}\left[\chi_6\mathrm{Im}(\phi^0\phi_{zz}^{0*})+\mathrm{Im}(\phi_\theta^0\phi^{0*})\right]\mathrm{d}S=-\frac{1}{2K}\mathrm{Re}\int_{S_F}(\phi_\beta^{0*}\phi_n^1-\phi^1\phi_{\beta n}^{0*})\mathrm{d}S. \tag{6.7}$$

We then apply Green's theorem to the potentials ϕ_β^{0*} and ϕ^1 and exploit the body boundary conditions for these potentials, i.e. $\phi_{\beta n}^{0*}=\phi_n^1=0$ at S_B. The integral over S_F on the right-hand side of (6.7) is then converted to an integral over $S(R)$ with the same integrand, but with the minus sign in front of the integral replaced by a plus sign. Upon introducing the result into equation (6.6) for B_{66}, we find

$$\frac{B_{66}}{\rho g A^2}=\frac{1}{2K}\int_{S(R)}\mathrm{Re}\left[(\phi_\beta^{0*}+\phi_\theta^{0*})\phi_n^1-\phi^{1*}(\phi_\beta^0+\phi_\theta^0)_n\right]\mathrm{d}S. \tag{6.8}$$

It is noted that the expression (6.8) for B_{66} agrees with Newman (1993, equation 5.10). Newman derived this expression by integrating the pressure over the wetted body surface, converting the resulting formulae to integrals over the control surface in the far field. In his approach, perturbation expansions of the potentials about a fixed position of the body is applied, assuming that the rotation angle and the angular velocity are both small. In our analysis B_{66} is obtained by applying conservation of angular momentum, allowing the rotation angle of the body to be finite, but assuming that the angular velocity is small. Thus, (6.8) being valid for a rotation angle of the body that is not small, is a generalization of Newman (1993).

We then introduce $\phi^1 = 2\mathrm{i}K(\phi^0_{\beta K} + \phi^0_{\theta K}) + \phi^{13}$ in (6.8) and note that the divergent parts of ϕ^1 cancel in the integral over $S(R)$ for $R \to \infty$. B_{66} may therefore be expressed in terms of the far-field amplitudes of ϕ_7 and ϕ^{13}, giving as the final result

$$\frac{B_{66}}{\rho g A^2} = \frac{1}{2K}\mathrm{Im}\int_0^{2\pi}[H_\beta^{0*}(\theta) + H_\theta^{0*}(\theta)]H^1(\theta)\mathrm{d}\theta \qquad (6.9)$$

where the amplitude function H^1 is introduced by

$$H^1 = 2\mathrm{i}K(H^0_{\beta K} + H^0_{\theta K}) + H^{13}; \qquad (6.10)$$

H^0 and H^{13} are given by (4.19) and (4.21), respectively.

7. The damping force

By applying conservation of linear momentum we obtain the following expression for the mean horizontal force, $\boldsymbol{F} = F_1\boldsymbol{i}_1 + F_2\boldsymbol{i}_2$:

$$F_i = \boldsymbol{i}_i\cdot\left(\rho\Omega\frac{\partial}{\partial\beta}\overline{\int_V \boldsymbol{v}'\mathrm{d}V} - \rho\boldsymbol{\Omega}\times\overline{\int_V \boldsymbol{v}'\mathrm{d}V} - \overline{\int_{S(R)}(p\boldsymbol{n} + \rho\boldsymbol{v}'\boldsymbol{v}\cdot\boldsymbol{n})\mathrm{d}S}\right), \quad i = 1,2. \qquad (7.1)$$

Using the same control surface as in the previous section, we may use in the last term that $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{v}'\cdot\boldsymbol{n}$. In the derivations below, terms proportional to Ω^2, A^3 are neglected. Consider first the pressure term in (7.1). By inserting (2.7) for the pressure we obtain

$$\overline{\int_{S(R)} -\frac{p}{\rho}n_i\mathrm{d}S} = -\frac{1}{2g}\int_{C(R)}\overline{(\Phi_t - \Omega\Phi_\theta)^2}n_i\mathrm{d}l + \int_{S(R)}(-\Omega\psi_\beta^{(2)} - \Omega\psi_\theta^{(2)} + \tfrac{1}{2}\overline{|\nabla\Phi|^2})n_i\mathrm{d}S \quad (7.2)$$

where we have exploited the fact that the surface elevation far away from the body is given by $\zeta = -(1/g)(\Phi_t - \Omega\Phi_\theta)$. By then introducing $\Phi = \mathrm{Re}[(A\mathrm{i}g/\omega)\phi\mathrm{e}^{\mathrm{i}\omega t}]$, and noting that $\partial\Phi/\partial t = \mathrm{Re}[(A\mathrm{i}g/\omega)(\mathrm{i}\omega\phi - \Omega\phi_\beta)\mathrm{e}^{\mathrm{i}\omega t}]$, we obtain

$$\overline{\int_{S(R)} -\frac{p}{\rho}n_i\mathrm{d}S} = -\frac{gA^2}{4}\mathrm{Re}\int_{C(R)}[|\phi|^2 - \frac{2\mathrm{i}\Omega}{\omega}\phi(\phi^*_\beta + \phi^*_\theta)]n_i\mathrm{d}l$$
$$+\frac{gA^2}{4K}\int_{S(R)}|\nabla\phi|^2 n_i\mathrm{d}S - \Omega\int_{S(R)}(\psi_\beta^{(2)} + \psi_\theta^{(2)})n_i\mathrm{d}S. \qquad (7.3)$$

Consider then $\overline{\int_V \boldsymbol{v}'\mathrm{d}V}$. By using Gauss' theorem this integral may be rewritten as integrals over S_B, S_F and $S(R)$. For the horizontal components we find

$$\boldsymbol{i}_i\cdot\overline{\int_V \boldsymbol{v}'\mathrm{d}V} = \int_{S_B+S(R)} n_i\psi^{(2)}\mathrm{d}S - \frac{A^2 g}{2\omega}\int_{S_F}\mathrm{Im}(\phi_{x_i}\phi^*)\mathrm{d}S, \quad i = 1,2. \qquad (7.4)$$

The integral over S_B in (7.4) may be rewritten by applying Green's theorem to χ_i, $i = 1,2$, introduced in §3.1, and $\psi^{(2)}$, giving

$$\int_{S_B} n_i\psi^{(2)}\mathrm{d}S = \int_{S_F}\chi_i\psi_n^{(2)}\mathrm{d}S = -\frac{A^2 g}{2\omega}\int_{S_F}\chi_i\mathrm{Im}(\phi\phi^*_{zz})\mathrm{d}S, \quad i = 1,2, \qquad (7.5)$$

where we have exploited the boundary conditions for $\psi^{(2)}$ and χ_i. Next we apply

Gauss' theorem to rewrite the integral over S_F in (7.4), i.e.

$$-\frac{A^2 g}{2\omega}\mathrm{Im}\int_{S_F}\phi_{x_i}\phi^* \mathrm{d}S = \frac{A^2 g}{2\omega}\mathrm{Im}\Big(\int_{S_F} x_i\phi\phi^*_{zz}\mathrm{d}S + \int_{C_B+C(R)} x_i\phi\phi^*_n \mathrm{d}l\Big). \tag{7.6}$$

In the diffraction problem $\partial\phi/\partial n = 0$ at C_B, provided that the body is wall-sided. By inserting (7.3)–(7.6) into (7.1) we find for the force

$$\begin{aligned}\frac{\boldsymbol{F}}{\rho g A^2} &= \frac{\epsilon}{2}\left(-\boldsymbol{i}_i\frac{\partial}{\partial\beta} + \boldsymbol{k}\times\boldsymbol{i}_i\right)\left(\int_{S_F}(\chi_i - x_i)\mathrm{Im}(\phi\phi^*_{zz})\mathrm{d}S - \int_{C(R)} x_i \mathrm{Im}(\phi\phi^*_n)\mathrm{d}l\right)\\ &\quad + \frac{\boldsymbol{i}_i}{4K}\mathrm{Re}\left(-K\int_{C(R)}\phi\phi^* n_i \mathrm{d}l + \int_{S(R)}(\nabla\phi\cdot\nabla\phi^* n_i - 2\phi_{x_i}\phi^*_n)\mathrm{d}S\right)\\ &\quad - \frac{\epsilon \boldsymbol{i}_i}{2}\int_{C(R)}\mathrm{Im}[\phi(\phi^*_\beta + \phi^*_\theta)]n_i \mathrm{d}l\end{aligned} \tag{7.7}$$

where the sum is over $i = 1, 2$. By applying a variation of Stokes' theorem we may show that, see (B 5) in Appendix B,

$$-\int_C K\phi\phi^* n_i \mathrm{d}l + \int_S[\nabla\phi\cdot\nabla\phi^* n_i - \phi_{x_i}\phi^*_n]\mathrm{d}S = \int_S \phi(\phi^*_{x_i})_n \mathrm{d}S - 2\mathrm{i}K\epsilon\int_C \phi(\phi^*_\beta + \phi^*_\theta)n_i \mathrm{d}l. \tag{7.8}$$

By combining (7.7) and (7.8) we find

$$\begin{aligned}\frac{\boldsymbol{F}}{\rho g A^2} &= \frac{\epsilon}{2}\left(-\boldsymbol{i}_i\frac{\partial}{\partial\beta} + \boldsymbol{k}\times\boldsymbol{i}_i\right)\left(\int_{S_F}(\chi_i - x_i)\mathrm{Im}(\phi\phi^*_{zz})\mathrm{d}S - \int_{C(R)} x_i \mathrm{Im}(\phi\phi^*_n)\mathrm{d}l\right)\\ &\quad + \frac{\boldsymbol{i}_i}{2K}\mathrm{Re}\int_{S(R)}[\phi(\phi^*_{x_i})_n - \phi_{x_i}\phi^*_n]\mathrm{d}S.\end{aligned} \tag{7.9}$$

We then introduce $\phi = \phi^0 + \epsilon\phi^1$, and expand the force in ϵ, i.e. $\boldsymbol{F} = \boldsymbol{F}_0 - \epsilon(B_{16}\boldsymbol{i} + B_{26}\boldsymbol{j})$, where a minus sign is adopted since $-\epsilon(B_{16}\boldsymbol{i} + B_{26}\boldsymbol{j})$ formally appears as a damping force, giving

$$\frac{\boldsymbol{F}_0}{\rho g A^2} = \frac{1}{4K}\mathrm{Re}\int_{S(R)}[\phi^0(\nabla_h\phi^{0*})_n - \phi^0_{x_i}\phi^{0*}_n]\mathrm{d}S, \tag{7.10}$$

$$\begin{aligned}\frac{B_{16}\boldsymbol{i} + B_{26}\boldsymbol{j}}{\rho g A^2} &= \mathrm{Im}\frac{1}{2}\left(\boldsymbol{i}_i\frac{\partial}{\partial\beta} - \boldsymbol{k}\times\boldsymbol{i}_i\right)\left(\int_{S_F}(\chi_i - x_i)\phi^0\phi^{0*}_{zz}\mathrm{d}S - \int_{C(R)} x_i\phi^0\phi^{0*}_n \mathrm{d}l\right)\\ &\quad -\mathrm{Re}\frac{1}{4K}\int_{S(R)}[\phi^0(\nabla_h\phi^{1*})_n + \phi^1(\nabla_h\phi^{0*})_n - \phi^0_n\nabla_h\phi^{1*} - \phi^1_n\nabla_h\phi^{0*}]\mathrm{d}S.\end{aligned} \tag{7.11}$$

The formula (7.11) is in a different form than that obtained by Newman (1993), and in addition to that we here allow the rotation angle to be finite. It is, however, possible by appropriate transformations to compare his and our results. We find that there are discrepancies between (7.11) and his final result (5.6). Thus, to Newman's equation (5.6) we must add a term $(\partial/\partial\beta)\begin{pmatrix}-2\pi\mu_1\\-2\pi\mu_2\end{pmatrix}$ and change the sign on his term $\begin{pmatrix}2\pi\mu_2\\-2\pi\mu_1\end{pmatrix}$ (in Newman's notation), to obtain our formulae. We note that $2\pi\mu_i = -\omega/(2g)\int_{S_F}(\chi_i - x_i)\mathrm{Im}(\phi^0\phi^{0*}_{zz})\mathrm{d}S$, $i = 1, 2$. However, our formula (7.11) may after some algebra be brought into a form which is equivalent to Newman's equation (5.5), which preceeds Newman's equation (5.6). To obtain this agreement we must disregard his comments following equation (5.5), where it is claimed that the second integral over S_F vanishes for $i = 1, 2$.

Further development of the integrals in (7.11) requires some algebra. This is outlined in Appendix C, where we find that B_{16} and B_{26} may be expressed in terms of the far-field amplitudes of the potentials ϕ^7, ϕ^{13}, and the dipole moments of the time-averaged second-order potential (with integrals over the free surface), with the final result

$$\frac{B_{16}\boldsymbol{i} + B_{26}\boldsymbol{j}}{\rho g A^2} = \frac{1}{2}\left(\boldsymbol{i}_i\frac{\partial}{\partial\beta} - \boldsymbol{k}\times\boldsymbol{i}_i\right)\int_{S_F}(\chi_i - x_i)\mathrm{Im}(\phi^0\phi^{0*}_{zz})\mathrm{d}S$$
$$+\frac{1}{2}\mathrm{Re}\int_0^{2\pi}\left[H^0(\theta)H^{1*}(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} - 2\mathrm{i}H^0(\theta)(H^{0*}_{\beta\theta}(\theta) + H^{0*}_{\theta\theta}(\theta))\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right]\mathrm{d}\theta$$
$$+\frac{1}{2}\mathrm{Re}\left\{\left(\frac{2\pi}{K}\right)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}\left[H^{1*}(\beta)\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix} - 2\mathrm{i}(H^{0*}_{\beta\theta}(\beta) + H^{0*}_{\theta\theta}(\beta))\begin{pmatrix}-\sin\beta\\ \cos\beta\end{pmatrix}\right]\right\}. \quad (7.12)$$

8. Conservation of energy

In the general case the energy equation reads

$$\frac{W}{\rho} = -\overline{\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left(\tfrac{1}{2}\boldsymbol{v}^2 + gz\right)\mathrm{d}V} - \overline{\int_{S(R)}\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{v}^2 + gz\right)\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S} \qquad (8.1)$$

where $W = \overline{\int_{S_B} p\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}S}$ denotes the mean work performed by the pressure force on the body. In the present example the body is restrained, and no work is performed, thus $W = 0$. Equation (8.1) may then be utilized as a check on the model and the computational procedure.

As in the previous sections, $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{v}'\cdot\boldsymbol{n}$ at $S(R)$. Furthermore we have

$$-\overline{\frac{\mathrm{d}}{\mathrm{d}t}\int_V\left(\tfrac{1}{2}\boldsymbol{v}^2 + gz\right)\mathrm{d}V} = \Omega\frac{\partial}{\partial\beta}\overline{\int_V\left(\tfrac{1}{2}\boldsymbol{v}^2 + gz\right)\mathrm{d}V}.$$

It may be shown that

$$\Omega\frac{\partial}{\partial\beta}\overline{\int_V\left(\tfrac{1}{2}\boldsymbol{v}^2 + gz\right)\mathrm{d}V} = \Omega\frac{\partial}{\partial\beta}\left(\frac{A^2g}{2}\int_{S_F}|\phi^0|^2\mathrm{d}S + \frac{A^2g}{4K}\mathrm{Re}\int_{S(R)}\phi^0\phi^{0*}_n\mathrm{d}S\right) \qquad (8.2)$$

where we have neglected terms proportional to Ω^2. Furthermore we have

$$-\overline{\int_{S(R)}\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{v}^2 + gz\right)\boldsymbol{v}'\cdot\boldsymbol{n}\,\mathrm{d}S}$$
$$= \int_{S(R)}\overline{\Phi'_t\Phi'_n}\mathrm{d}S = \frac{A^2g^2}{2\omega^2}\mathrm{Re}\int_{S(R)}(\mathrm{i}\omega\phi - \Omega\phi_\beta)\phi^*_n\mathrm{d}S. \qquad (8.3)$$

By introducing $\phi = \phi^0 + \epsilon\phi^1$ and expanding the right-hand side of (8.1) in $W^0 + \epsilon W^1$, we obtain

$$\frac{W^0}{\rho g A^2 c_g} = -\mathrm{Im}\int_{S(R)}\phi^0\phi^{0*}_n\mathrm{d}S, \qquad (8.4)$$

$$\frac{W^1}{\rho g A^2 c_g} = \frac{\partial}{\partial\beta}\left(K\int_{S_F}|\phi^0|^2\mathrm{d}S + \mathrm{Re}\frac{1}{2}\int_{S(R)}\phi^0\phi^{0*}_n\mathrm{d}S\right)$$
$$-\mathrm{Im}\int_{S(R)}(\phi^1\phi^{0*}_n + \phi^0\phi^{1*}_n + \mathrm{i}\phi^0_\beta\phi^{0*}_n)\mathrm{d}S \qquad (8.5)$$

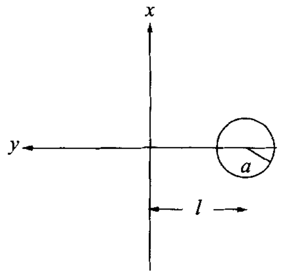

FIGURE 1. Sketch of the location of a vertical circular cylinder with radius a and draught $3a$. The axis is located at $x = 0$, $y = -l$.

where we have introduced the group velocity of the waves, $c_g \equiv \partial\omega/\partial K = g/2\omega$.

By then applying $\phi^1 = \phi^{11} + \phi^{12} + \phi^{13}$, we may after some algebra obtain that

$$\frac{W^1}{\rho g A^2 c_g} = -2K\frac{\partial}{\partial K}\mathrm{Re}\int_{S(R)} \phi_\theta^0 \phi_n^{0*}\mathrm{d}S - \mathrm{Im}\int_{S(R)} (\phi^{13}\phi_n^{0*} + \phi^0\phi_n^{13*})\mathrm{d}S. \tag{8.6}$$

Substituting for ϕ^0 and ϕ^{13}, and letting $R \to \infty$, we obtain

$$\frac{W^1}{\rho g A^2 c_g} = -K\frac{\partial}{\partial K}\mathrm{Im}\left(\int_0^{2\pi} H^0(\theta)H_\theta^0(\theta)^*\mathrm{d}\theta + \left(\frac{8\pi}{K}\right)^{1/2} H_\theta^0(\beta)^*\mathrm{e}^{\mathrm{i}\pi/4}\right)$$
$$- \mathrm{Re}\left(\int_0^{2\pi} H^0(\theta)H^{13}(\theta)^*\mathrm{d}\theta + \left(\frac{2\pi}{K}\right)^{1/2} H^{13}(\beta)^*\mathrm{e}^{\mathrm{i}\pi/4}\right) \tag{8.7}$$

where the method of stationary phase is applied to obtain the terms containing $H_\theta^0(\beta)$ and $H^{13}(\beta)$.

9. Numerical results

Results for B_{i6}, $i = 1, 2, 6$, are considered for two different geometries. In the first example the body is a vertical circular cylinder with radius a, draught $3a$, and located with its axis at $x = 0$, $y = -l$, where l is arbitrary, see figure 1. In the absolute frame of reference the cylinder axis describes a circular path about the origin with radius l. Since the body is a vertical circular cylinder, the moment with respect to the z-axis equals the force component along the x-axis multiplied by the arm l, i.e. $M_{z0} = l\boldsymbol{F}_0 \cdot \boldsymbol{i}$, $B_{66} = lB_{16}$. In figure 2 are shown results for the damping coefficients B_{16} normalized by $\rho\omega^2 aA^2 l$ and B_{66} normalized by $\rho\omega^2 aA^2 l^2$ for three different wave angles. The numerical results confirm that B_{16}/l and B_{66}/l^2 are independent of the value of l, and that $B_{66} = lB_{16}$, which are both expected results. We remark that the variation of the damping coefficients B_{16} and B_{66} with respect to the wave angle, keeping the wavenumber fixed, satisfies the following relations for this example: $B_{16}(\pi/2) \leqslant B_{16}(\beta) \leqslant B_{16}(\pi)$ and $B_{66}(\pi/2) \leqslant B_{66}(\beta) \leqslant B_{66}(\pi)$ for all β.

In figure 3 is shown B_{26} for wave angle $\beta = 3\pi/4$. B_{26} is always negative or zero. The numerical results confirm, as expected, that B_{26}/l is independent of the value of l. For this geometry, it is noted that B_{16}/l, B_{26}/l, and B_{66}/l^2 may be obtained from the wave drift damping coefficients due to a vertical cylinder translating along the positive x-direction. Indeed, the results shown in figures 2 and 3 are in exellent agreement with the corresponding wave drift damping coefficients using the method of Nossen *et al.* (1991) for the translatory case.

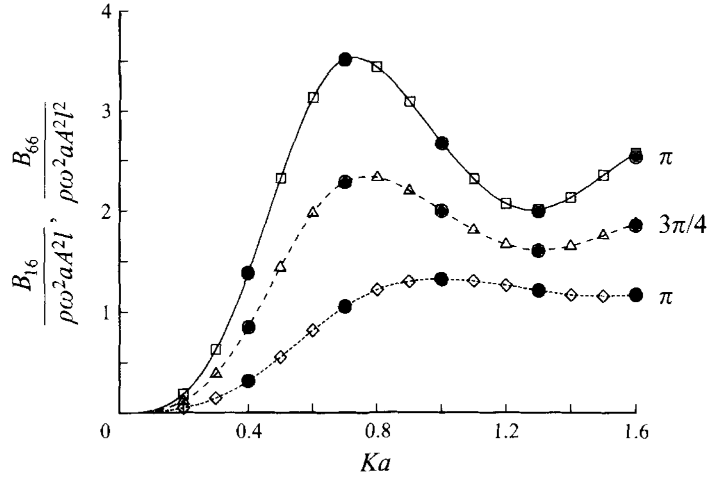

FIGURE 2. Numerical results for $B_{16}/\rho\omega^2 aA^2 l$ and $B_{66}/\rho\omega^2 aA^2 l^2$ *vs.* Ka for the vertical cylinder described in figure 1, and three different wave headings. Wave amplitude A, wavenumber K. $B_{16}/\rho g A^2 l$: solid line, $\beta = \pi$; dashed line, $\beta = 3\pi/4$; dotted line, $\beta = \pi/2$. $B_{66}/\rho\omega^2 aA^2 l^2$: squares, $\beta = \pi$; triangles, $\beta = 3\pi/4$; diamonds, $\beta = \pi/2$. Black circles: results for a cylinder in translatory motion, obtained by the method of Nossen *et al.* (1991). Discretization: S_B 784 panels, S_F 784 panels. S_F is discretized out to the circle with centre in the cylinder axis and radius $7a$.

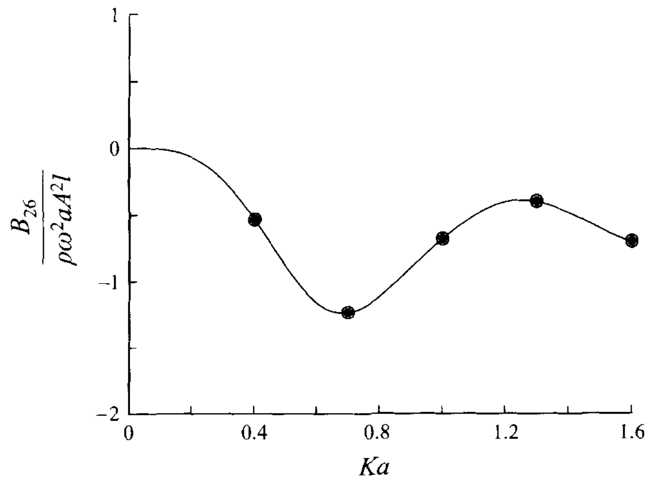

FIGURE 3. Numerical results for $B_{26}/\rho\omega^2 aA^2 l$ *vs.* Ka for the vertical cylinder described in figure 1. Black circles: Results for a translatory cylinder obtained using the method of Nossen *et al.* (1991). $\beta = 3\pi/4$. $B_{26} = 0$ for $\beta = \pi/2$ and $\beta = \pi$. Discretization: S_B 784 panels, S_F 784 panels. S_F is discretized out to the circle with centre in the cylinder axis and radius $7a$.

We have also invoked the energy equation for the vertical cylinder, see (8.1), (8.4), (8.7), which in our case predict that $W = 0$, since the body performs no work on the fluid. In all computations we find that $W^0 \simeq 0$ and $W^1 \simeq 0$. For example, in computing W^1, we find that both terms on the right-hand side of (8.7) are large, but cancel each other almost exactly, see figure 4.

In the next example the geometry is a ship, length l and beam b_0, with $l/b_0 = 5.6$. The ship section is a circular half-cylinder with radius $r(x) = 0.5b_0[1-(2x/l)^4]$, $|x| \leqslant l/2$. Numerical values of B_{66} are shown in figure 5 for $\beta = \pi/2$ (beam seas),

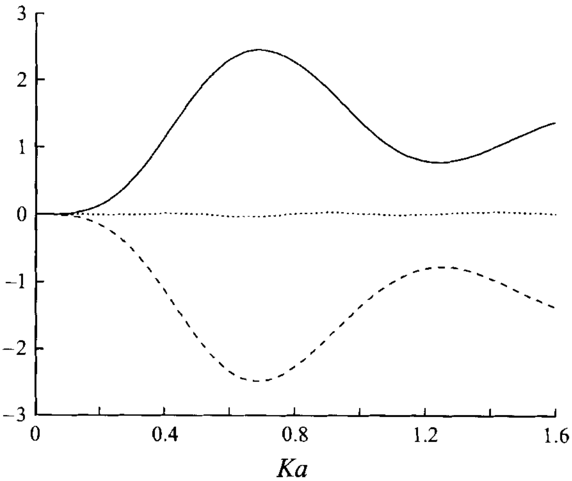

FIGURE 4. Energy equation *vs.* Ka for the vertical cylinder described in figure 1. $\beta = \pi$. Solid line the first term and dashed line the second term on the right-hand side of (8.7) divided by l. Dotted line: $W^1/\rho g A^2 c_g l$.

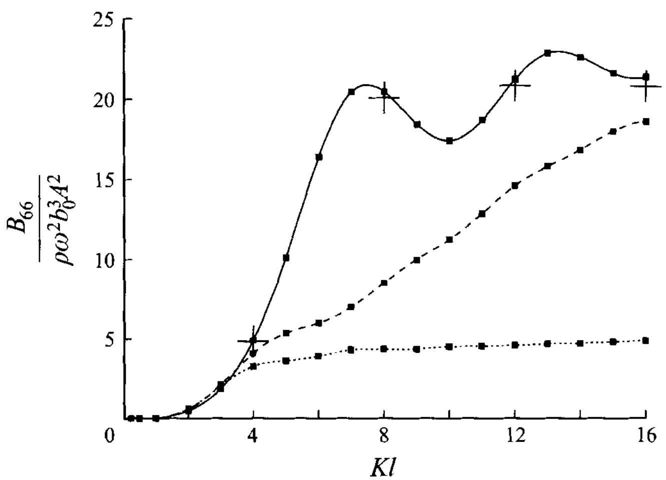

FIGURE 5. $B_{66}/\rho\omega^2 b_0^3 A^2$ *vs.* Kl for the ship. Solid line: $\beta = \pi/2$; dashed line: $\beta = 3\pi/4$. Dotted line: $\beta = \pi$. The black squares denote the computation points. S_B and S_F are both discretized with 800 panels. S_F is discretized out to a circle with radius l and centre in the origin. Crosses mark computations with panelization: S_B and S_F 1568 panels.

$\beta = 3\pi/4$ (quartering seas), $\beta = \pi$ (head seas). B_{66} is always positive, is largest for beam seas, and smallest for head seas. More specifically, B_{66} is 4–5 times larger for beam seas than for head seas, when $Kl > 6$.

In figures 6 and 7 are displayed results for B_{16} and B_{26} for the ship. B_{16} and B_{26} attain both positive and negative values. For long waves we find that $B_{16}(\pi/2) = -B_{26}(\pi)$, $B_{16}(\beta) = B_{16}(\pi/2)\sin\beta$, and $B_{26}(\beta) = B_{26}(\pi)\cos(\pi - \beta)$. It is noted that B_{16} for beam seas and B_{26} for head seas both are large. These results correspond to non-vanishing moments with respect to the z-axis, due to a ship translating respectively along the x-axis in beam seas, and along the y-axis in head seas, see Grue & Palm (1993).

We have assumed in the theory that $\omega \gg \Omega$. This means that the values of the

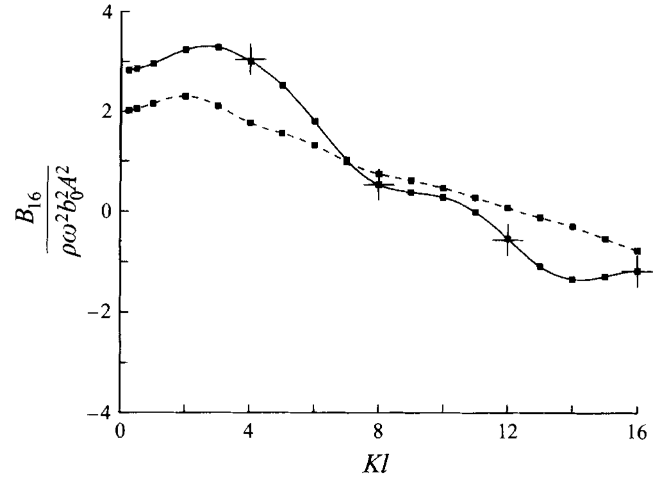

FIGURE 6. As figure 5 but for $B_{16}/\rho\omega^2 b_0^2 A^2$. $B_{16} = 0$ for $\beta = \pi$.

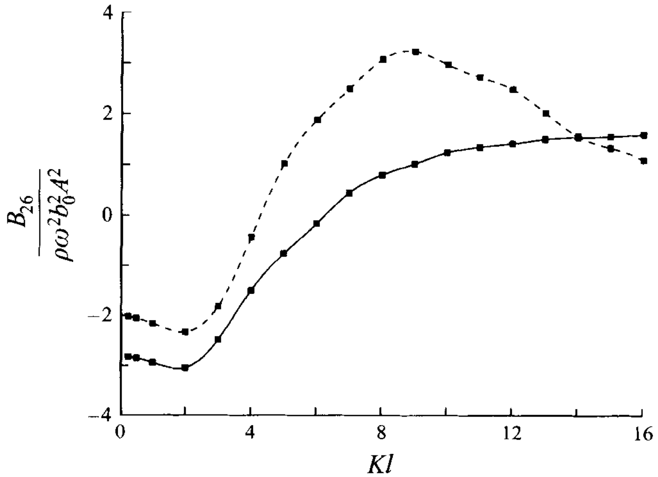

FIGURE 7. As figure 5 but for $B_{26}/\rho\omega^2 b_0^2 A^2$. Solid line: $\beta = \pi$. Dashed line: $\beta = 3\pi/4$. $B_{26} = 0$ for $\beta = \pi/2$.

damping coefficients shown in the figures for small Kl are not relevant to the slow drift problem, strictly speaking. It is still, however, of interest to investigate the results in the limit $Kl \to 0$.

9.1. *Convergence*

We next investigate convergence of the method. The geometry is modelled by a set of quadrilaterals and the potentials and the source strengths are approximated by constants at each quadrilateral. The numerical integration is performed by the midpoint rule, except when integrating the singularities of the Green function. Thus, we expect that the relative error in the integrated forces and moments is of the order of a typical panel size divided by the wetted area of the body, i.e. proportional to $1/N_B$, $1/N_F$, where N_B and N_F denote the number of panels on S_B and S_F, respectively. In table 1(a) are shown results for B_{66} and in table 1(b) for B_{16} for various discretizations of the ship. The results in the table clearly indicate convergence as the number of

	N_B	N_F	$Kl=4$	$Kl=8$	$Kl=12$	$Kl=16$
(*a*)	392	392	5.000	21.20	22.01	22.51
	800	800	4.919	20.47	21.27	21.41
	1568	1568	4.869	20.11	20.88	20.84
(*b*)	392	392	−2.940	−0.509	0.517	1.232
	800	800	−3.009	−0.523	0.537	1.160
	1568	1568	−3.049	−0.526	0.566	1.171

TABLE 1. Convergence of (*a*) $B_{66}/\rho\omega^2 b_0^3 A^2$ (*b*) $B_{16}/\rho\omega^2 b_0^2 A^2$ for the ship *vs.* panelization. Free surface discretized within a circle with radius l. N_B denotes number of panels on S_B, N_F denotes number of panels on S_F. $\beta = \pi/2$.

	R_{out}	N_F	$Kl=4$	$Kl=8$	$Kl=12$	$Kl=16$
(*a*)	no S_F	0	8.484	39.87	49.48	54.65
	$0.75l$	440	4.976	20.56	21.34	21.42
	l	800	4.919	20.47	21.27	21.41
	$1.5l$	1800	4.882	20.27	21.01	21.16
	$2l$	3200	4.848	20.10	20.82	20.97
(*b*)	no S_F	0	−0.402	0.340	2.177	3.758
	$0.75l$	440	−2.989	−0.393	0.508	1.174
	l	800	−3.009	−0.523	0.537	1.160
	$1.5l$	1800	−3.086	−0.486	0.508	1.175
	$2l$	3200	−3.181	−0.524	0.510	1.210

TABLE 2. Convergence of (*a*) $B_{66}/\rho\omega^2 b_0^3 A^2$ (*b*) $B_{16}/\rho\omega^2 b_0^2 A^2$ for the ship *vs.* discretization radius R_{out} of the free surface. $3l/4 < R_{out} < 2l$. Number of panels on S_B is in all cases $N_B = 800$. N_F denotes number of panels on S_F. $\beta = \pi/2$.

panels is increased. The results for B_{66} exhibit a linear convergence rate with respect to $1/N_B$, $1/N_F$. The convergence rate for B_{16} is somewhat different than for B_{66}. However, the difference between the finest and next finest discretization is minor. This is also seen from the plots in figures 5 and 6.

In the computations for the ship a truncation radius equal to the ship length was used. In tables 2(*a*) and 2(*b*) we investigate how the values of B_{66} and B_{16} depend on the truncation radius. We find that the computed values vary by at most 2% for B_{66} and at most 6% for B_{16} when the truncation radius is in the interval $l <$ truncation radius $< 2l$. The table also shows that integration over the free surface cannot be omitted.

9.2. *Comparison with viscous damping*

It is of interest to compare the wave drift damping moment of the ship with an estimate of the damping moment due to viscous drag. The latter may be obtained from the sectionwise drag on the ship which is given by $\mathrm{d}D = (1/2)\rho C_D r(x) x|x|\Omega|\Omega|\mathrm{d}x$ (with the x-axis in the length-direction of the ship). The yaw moment due to viscous drag is then obtained as

$$M_{visc} = \int_{-l/2}^{l/2} x\mathrm{d}D = \frac{1}{256}\rho C_D \Omega|\Omega| b_0 l^4. \tag{9.1}$$

Consider a practical case concerning the slow yaw motions of a moored Turret Production Ship described in Faltinsen (1990, p. 280). The data in this case are: ship length $l = 230$ m, natural period of the yaw motion 400 s, standard deviation of the

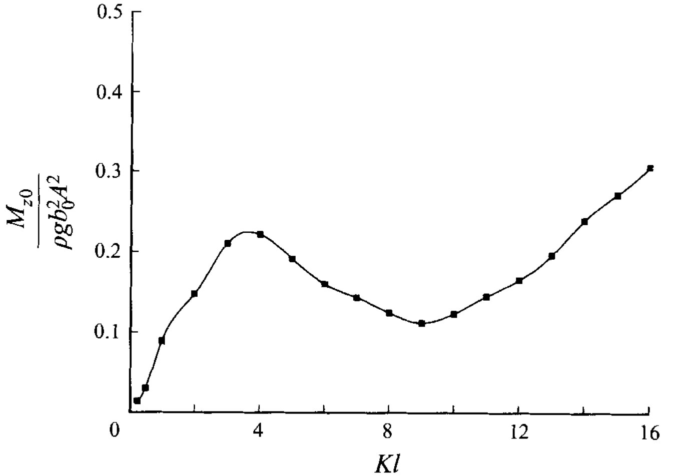

FIGURE 8. $M_{z0}/\rho g b_0^2 A^2$ *vs.* Kl for the ship. $\beta = 3\pi/4$. Same discretization as in figure 5.

slow yaw angle 3°. This means that the standard deviation of Ω is $8.22 \times 10^{-4}\mathrm{s}^{-1}$. The drag coefficient may be estimated as $C_D = 1$ (see Faltinsen 1990, p. 194). From the results in figure 5 we have that $B_{66}/\rho\omega^2 b_0^3 A^2 > 4$ for $\omega^2 l/g > 5$. (The wave period is less than 14 s when $\omega^2 l/g > 5$ and $l = 230$ m.) This means that

$$\frac{(\Omega/\omega)B_{66}}{M_{visc}} > \left(\frac{A}{1.7\ \mathrm{m}}\right)^2. \tag{9.2}$$

Thus, for wave amplitude $A > 1.7$ m the wave drift damping moment is larger than the damping moment due to viscous drag in this example.

9.3. *Comparison with the moment at $\Omega = 0$*

It is of interest to compare the magnitude of B_{66} to the zero-speed moment M_{z0}, where the latter is displayed in figure 8 for $\beta = 3\pi/4$. We observe that B_{66} is one order of magnitude larger that M_{z0}. For example, for $\beta = 3\pi/4$ and $Kl = 10$, we have $B_{66}/M_{z0} \simeq 200$. By applying $\Omega = 8.22 \times 10^{-4}\mathrm{s}^{-1}$ as in the example above, we find that $(\Omega/\omega)B_{66}$ is 25% of M_{z0}. This shows that the wave drift damping moment also significantly contributes to the total moment acting on the ship.

10. Concluding remarks

A method for evaluating the diffracted waves, the wave forces and the wave drift damping due to a floating body performing a slow rotation about the vertical axis, with a finite rotation angle, is developed. The incoming waves, with wave frequency ω in the absolute frame of reference, and the slow angular velocity Ω of the body, where it is assumed that $\Omega \ll \omega$, introduce two timescales to the problem, proportional to $1/\omega$ and $1/\Omega$, respectively. It is noted that the components of the exciting force experienced by the body, being proportional to the wave amplitude, generally will oscillate with a frequency $\omega + O(\Omega)$, which is slightly different from ω and varying with respect to the wave angle. The corresponding result is true for the linear fluid flow at a fixed space location.

The mathematical problem is formulated by means of potential theory. A set of boundary value problems is developed by applying perturbation expansions in the incoming wave amplitude and the slow angular velocity of the body, which are solved by means of integral equations, containing unknown quantities on the wetted body surface only. It is noted that the boundary value problems and the corresponding integral equations due to a slowly translating body are obtained as part results of the present analysis.

Formulae for the wave drift damping coefficients are obtained by applying conservation of linear and angular momentum. The resulting expression for the damping coefficient B_{66} given by (6.8) has formally the same form as Newman (1993, equation 5.10), except that β_0 (the wave angle in the fixed frame of reference) is replaced by $\beta = \beta_0 - \alpha$ in our formula (β the wave angle in the relative frame of reference, α the rotation angle of the body in the fixed frame of reference). In Newman's analysis the rotation angle is assumed small, and hence his result is recovered from ours by setting $\alpha = 0$.

There are, however, disagreements between our formulae for B_{16} and B_{26} and the corresponding formulae obtained by Newman (1993, equation 5.6). To compare them we must perform some transformations. We then find that to get agreement we must add to Newman's equation (5.6) a term and change the sign of another term, as discussed in §7.

Here we proceed one step further than Newman (1993), as the wave drift damping coefficients are expressed in terms of the far-field amplitudes of the wave potentials and the dipole moments of the time-averaged second-order potential, resulting in simple formulae that are suitable for efficient numerical algoritms. Numerical solution of the problem is obtained by using a low-order panel method. The method is applicable to geometries of general form.

Examples for the damping coefficients B_{16}, B_{26}, B_{66}, are considered for two different bodies, namely a symmetric ship and a vertical circular cylinder with axis describing a circular path in the horizontal plane. The method is carefully checked, for example by invoking the balance of energy, which in all examples is satisfied to a relative accuracy of better than 1%. Convergence of the method is documented. We find that the damping coefficients are one order of magnitude larger than the time-averaged horizontal force components F_{x0}, F_{y0}, and the vertical moment M_{z0}, which means that a slow rotation of the body introduces a significant change of the forces. The damping coefficient B_{66} is in all the present examples found to be positive and large. This means that wave drift damping due to a slow yaw motion of the body is just as pronounced as for slow translatory motions. Thus, evaluation of the complete wave drift damping matrix (1.2) is required to study slow drift motions of moored floating bodies in the realistic manner.

We have also compared the wave drift damping to viscous damping. In the realistic case of a ship we find that the wave drift damping predominates even for waves with relatively small amplitude.

In the present contribution only the linear diffraction effects are taken into account, which means that the body is kept fixed in the relative frame of reference. The method may, however, be generalized to account for the linear responses of the floating body and the resulting radiation effects. The complete diffraction–radiation problem is under development, see Finne & Grue (1995). Relevant to most practical examples where wave drift damping is of importance, we have considered the water depth to be infinite. This has also simplified the analysis.

We gratefully acknowledge the financial support by Conoco Norway, Det Norske Veritas, Saga Petroleum, and Statoil. The wave radiation-diffraction program WAMIT was provided by Massachusetts Institute of Technology and Det Norske Veritas.

Appendix A. The integral in (4.13)

The integral (4.13) is given by

$$I = \int_{S(R)} \left(\phi^{12} \frac{\partial G^0}{\partial n} - G^0 \frac{\partial \phi^{12}}{\partial n} - \phi^0 \frac{\partial G^1}{\partial n} + G^1 \frac{\partial \phi^0}{\partial n} \right) \mathrm{d}S. \tag{A 1}$$

By applying Green's theorem to G^0 and $2\mathrm{i}K\partial^2\phi_I/\partial\beta\partial K$, and to G^1 and ϕ_I, and integrating over the entire free surface and $S(R)$, we obtain

$$\int_{S_F+S(R)} \left(2\mathrm{i}K \frac{\partial^2 \phi_I}{\partial\beta\partial K} \frac{\partial G^0}{\partial n} - 2\mathrm{i}K G^0 \frac{\partial}{\partial n} \frac{\partial^2 \phi_I}{\partial\beta\partial K} + \phi_I \frac{\partial G^1}{\partial n} - G^1 \frac{\partial \phi_I}{\partial n} \right) \mathrm{d}S + 8\pi\mathrm{i}K \frac{\partial^2 \phi_I}{\partial\beta\partial K} = 0. \tag{A 2}$$

Applying $\partial\phi_I/\partial\beta = -\partial\phi_I/\partial\theta$, using (3.22) for ϕ^{12}, and (4.8) for G^1, and noting that the integral over the free surface in (A 2) vanishes, we obtain by adding (A 2) to the right-hand side of (A 1)

$$I = \int_{S(R)} \left(2\mathrm{i}K \frac{\partial^2 \phi_7}{\partial\theta\partial K} \frac{\partial G^0}{\partial n} - G^0 2\mathrm{i}K \frac{\partial}{\partial n} \frac{\partial^2 \phi_7}{\partial\theta\partial K} - \phi_7 \frac{\partial G^1}{\partial n} + G^1 \frac{\partial \phi_7}{\partial n} \right) \mathrm{d}S + 8\pi\mathrm{i}K \frac{\partial^2 \phi_I}{\partial\beta\partial K}. \tag{A 3}$$

By then applying partial integration with respect to the θ-variable, (A 3) reduces to

$$I = 2\mathrm{i}K \frac{\partial}{\partial K} \int_{S(R)} \left(\frac{\partial \phi_7}{\partial\theta} \frac{\partial G^0}{\partial R} - G^0 \frac{\partial^2 \phi_7}{\partial\theta\partial R} \right) \mathrm{d}S + 8\pi\mathrm{i}K \frac{\partial^2 \phi_I}{\partial\beta\partial K}. \tag{A 4}$$

It may be shown by using Green's theorem that the integral over $S(R)$ in (A 4) is independent of the value of R ($R >$body radius), and equals zero, which is obtained by letting $R \to \infty$ and applying the radiation conditions for ϕ_7 and G^0. Thus,

$$I = 8\pi\mathrm{i}K \frac{\partial^2 \phi_I}{\partial\beta\partial K}. \tag{A 5}$$

Appendix B. A variant of Stokes' theorem

A variant of Stokes' theorem reads

$$\int_S (\boldsymbol{n} \times \nabla) \times (f\boldsymbol{V})\mathrm{d}S = \int_C \mathrm{d}\boldsymbol{l} \times (f\boldsymbol{V}), \tag{B 1}$$

where $\boldsymbol{V}$ denotes a differentiable vector field, f a differentiable scalar variable, and S an open surface bounded by the contour C. The line integral around C is oriented positive with respect to the normal vector of the surface S. In our applications, C is a contour in the mean free surface, and S is assumed to have vertical walls at the mean free surface. We note that

$$(\boldsymbol{n} \times \nabla) \times (f\boldsymbol{V}) = f\boldsymbol{n} \times (\nabla \times \boldsymbol{V}) + f(\boldsymbol{n}\cdot\nabla)\boldsymbol{V} - \boldsymbol{n}\nabla\cdot(f\boldsymbol{V}) + (\boldsymbol{n}\cdot\boldsymbol{V})\nabla f. \tag{B 2}$$

Applying $\boldsymbol{V} = \nabla\psi$ we have $\nabla \times \boldsymbol{V} = 0$. Noting that $\mathrm{d}\boldsymbol{l} \times (f\boldsymbol{V}) = \mathrm{d}l[f(\boldsymbol{k}\psi_n - \boldsymbol{n}\psi_z)]$ we then obtain

$$-\int_S f(\boldsymbol{n}\cdot\nabla)\nabla\psi\mathrm{d}S = -\int_S [\boldsymbol{n}\nabla\cdot(f\nabla\psi) - \psi_n\nabla f]\mathrm{d}S - \int_C [f(\boldsymbol{k}\psi_n - \boldsymbol{n}\psi_z)]\mathrm{d}l. \tag{B 3}$$

Let $f = \phi$, and $\psi = \phi^*$, where ϕ^* satisfies $\nabla^2\phi^* = 0$ in the fluid domain. Furthermore, let $C = C(R)$, and $S = S(R) + S(z)$, where $S(z)$ denotes a horizontal bottom with vertical coordinate $z \to -\infty$. Assuming no fluid motion at $z = -\infty$, the contributions due to integrals over $S(z)$ vanish. Exploiting the free-surface boundary condition (3.5) for ϕ, for a sufficiently large R, such that the effect of χ_6 is negligible, i.e.

$$\phi_z = K\phi + 2\mathrm{i}\epsilon K\frac{\partial\phi}{\partial\alpha} - 2\mathrm{i}\epsilon K\frac{\partial\phi}{\partial\theta}, \tag{B 4}$$

we find that the horizontal components of (B 3) read

$$-\int_C K\phi\phi^* n_i \mathrm{d}l + \int_S [\nabla\phi\cdot\nabla\phi^* n_i - \phi_{x_i}\phi_n^*]\mathrm{d}S = \int_S \phi(\phi_{x_i}^*)_n \mathrm{d}S - 2\mathrm{i}K\epsilon\int_C \phi(\phi_\beta^* + \phi_\theta^*)n_i\mathrm{d}l. \tag{B 5}$$

Appendix C. The forces

We shall evaluate the damping coefficients B_{16}, B_{26} which are given by (7.11), i.e.

$$\frac{B_{16}\boldsymbol{i} + B_{26}\boldsymbol{j}}{\rho g A^2} = \mathrm{Im}\frac{1}{2}\left(\boldsymbol{i}_i\frac{\partial}{\partial\beta} - \boldsymbol{k}\times\boldsymbol{i}_i\right)\left(\int_{S_F}(\chi_i - x_i)\phi^0\phi_{zz}^{0*}\mathrm{d}S - \int_{C(R)} x_i\phi^0\phi_n^{0*}\mathrm{d}l\right)$$
$$-\mathrm{Re}\frac{1}{4K}\int_{S(R)}[\phi^0(\nabla_h\phi^{1*})_n + \phi^1(\nabla_h\phi^{0*})_n - \phi_n^0\nabla_h\phi^{1*} - \phi_n^1\nabla_h\phi^{0*}]\mathrm{d}S. \tag{C 1}$$

Now, ϕ_7 and ϕ^{13} have the following asymptotic forms for large KR:

$$\phi_7 = R^{-1/2}H^0(\theta)\mathrm{e}^{Kz-\mathrm{i}KR}(1 + O((KR)^{-1})), \tag{C 2}$$
$$\phi^{13} = R^{-1/2}H^{13}(\theta)\mathrm{e}^{Kz-\mathrm{i}KR}(1 + O((KR)^{-1})), \tag{C 3}$$

where H^0 and H^{13} are given by (4.19) and (4.21), respectively. The potential ϕ^1 is given by

$$\phi^1 = 2\mathrm{i}K\frac{\partial}{\partial K}\left(\frac{\partial\phi_7}{\partial\beta} + \frac{\partial\phi_7}{\partial\theta}\right) + \phi^{13} \tag{C 4}$$

since $\partial\phi_I/\partial\beta + \partial\phi_I/\partial\theta = 0$. For large KR it is convenient to decompose ϕ^1 as

$$\phi^1 = \psi^1 + \varphi^1 \tag{C 5}$$

where

$$\psi^1 = 2KR^{1/2}(H_\beta^0 + H_\theta^0)\mathrm{e}^{Kz-\mathrm{i}KR}(1 + O((KR)^{-1})) \tag{C 6}$$

and

$$\varphi^1 = R^{-1/2}[H^1 + 2\mathrm{i}Kz(H_\beta^0 + H_\theta^0)]\mathrm{e}^{Kz-\mathrm{i}KR}(1 + O((KR)^{-1})). \tag{C 7}$$

Here, the amplitude H^1 is defined by

$$H^1 = 2\mathrm{i}K(H_{\beta K}^0 + H_{\theta K}^0) + H^{13}. \tag{C 8}$$

Introducing $\boldsymbol{n}(\theta) = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta$, $\boldsymbol{t}(\theta) = \mathrm{d}\boldsymbol{n}/\mathrm{d}\theta$, we have the following relations for the derivatives of the potentials:

$$\phi_{I,n} = -\mathrm{i}K\cos(\theta - \beta)\phi_I, \tag{C 9}$$
$$\nabla_h\phi_I = -\mathrm{i}K\phi_I\boldsymbol{n}(\beta), \tag{C 10}$$
$$(\nabla_h\phi_I)_n = -K^2\cos(\theta - \beta)\phi_I\boldsymbol{n}(\beta), \tag{C 11}$$
$$\phi_{7,n} = \left(-\mathrm{i}K - \frac{1}{2R}\right)\phi_7 + O((KR)^{-5/2}), \tag{C 12}$$

$$\nabla_h\phi_7 = \left(\left(-\mathrm{i}K - \frac{1}{2R}\right)\boldsymbol{n}(\theta) + \frac{1}{R}\frac{H^0_\theta}{H^0}\boldsymbol{t}(\theta)\right)\phi_7 + O((KR)^{-5/2}), \tag{C 13}$$

$$(\nabla_h\phi_7)_n = \left(\left(-K^2 + \frac{\mathrm{i}K}{R}\right)\boldsymbol{n}(\theta) - \frac{\mathrm{i}K}{R}\frac{H^0_\theta}{H^0}\boldsymbol{t}(\theta)\right)\phi_7 + O((KR)^{-5/2}), \tag{C 14}$$

$$\varphi^1_n = -\mathrm{i}K\varphi^1 + O((KR)^{-3/2}), \tag{C 15}$$

$$\nabla_h\varphi^1 = -\mathrm{i}K\varphi^1\boldsymbol{n}(\theta) + O((KR)^{-3/2}), \tag{C 16}$$

$$(\nabla_h\varphi^1)_n = -K^2\varphi^1\boldsymbol{n}(\theta) + O((KR)^{-3/2}), \tag{C 17}$$

$$\psi^1_n = \left(-\mathrm{i}K + \frac{1}{2R}\right)\psi^1 + O((KR)^{-3/2}), \tag{C 18}$$

$$\nabla_h\psi^1 = \left[\left(-\mathrm{i}K + \frac{1}{2R}\right)\psi^1\boldsymbol{n}(\theta) + \frac{1}{R}\psi^1_\theta\boldsymbol{t}(\theta)\right] + O((KR)^{-3/2}), \tag{C 19}$$

$$(\nabla_h\psi^1)_n = \left[\left(-K^2 - \frac{\mathrm{i}K}{R}\right)\psi^1\boldsymbol{n}(\theta) - \frac{\mathrm{i}K}{R}\psi^1_\theta\boldsymbol{t}(\theta)\right] + O((KR)^{-3/2}). \tag{C 20}$$

Consider now the integral

$$I(\phi^0,\phi^1) = -\frac{1}{4K}\mathrm{Re}\int_{S(R)}[\phi^0(\nabla_h\phi^{1*})_n + \phi^1(\nabla_h\phi^{0*})_n - \phi^0_n\nabla_h\phi^{1*} - \phi^1_n\nabla_h\phi^{0*}]\mathrm{d}S. \tag{C 21}$$

Now, $I(\phi^0,\phi^1) = I(\phi^0,\varphi^1) + I(\phi^0,\psi^1)$. First $I(\phi^0,\varphi^1)$ is evaluated. By introducing (C 9)–(C 17) into (C 21) we obtain

$$I(\phi^0,\varphi^1) = \mathrm{Re}\int_{S(R)} K[\phi_7\boldsymbol{n}(\theta) + \tfrac{1}{4}\phi_I(1+\cos(\theta-\beta))(\boldsymbol{n}(\beta)+\boldsymbol{n}(\theta))]\varphi^{1*}\mathrm{d}S + O((KR)^{-1}). \tag{C 22}$$

By carring out the vertical integration, noting that $\int_{-\infty}^0 \mathrm{e}^{2Kz}\mathrm{d}z = 1/(2K)$, $\int_{-\infty}^0 z\mathrm{e}^{2Kz}\mathrm{d}z = -1/(4K^2)$, we obtain

$$I(\phi^0,\varphi^1) = \tfrac{1}{2}\mathrm{Re}\int_0^{2\pi} H^0[H^{1*} + \mathrm{i}(H^0_\beta + H^0_\theta)^*]\boldsymbol{n}(\theta)\mathrm{d}\theta + \tfrac{1}{8}\mathrm{Re}\int_0^{2\pi} R^{1/2}\mathrm{e}^{\mathrm{i}KR(1-\cos(\theta-\beta))} \times(1+\cos(\theta-\beta))(\boldsymbol{n}(\beta)+\boldsymbol{n}(\theta))[H^{1*} + \mathrm{i}(H^0_\beta + H^0_\theta)^*]\mathrm{d}\theta \tag{C 23}$$

where we have neglected terms of $O((KR)^{-1})$. By applying the method of stationary phase to the last integral, and letting $KR \to \infty$, we obtain

$$I(\phi^0,\varphi^1) = \frac{1}{2}\mathrm{Re}\left(\int_0^{2\pi} H^0H^{1*}\boldsymbol{n}(\theta)\mathrm{d}\theta + \left(\frac{2\pi}{K}\right)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}H^{1*}\boldsymbol{n}(\beta)\right) + I_1, \tag{C 24}$$

where I_1 is expressed in terms of ϕ_I, ϕ_7, and ψ^1, i.e.

$$I_1 = \frac{1}{4K}\mathrm{Re}\int_0^{2\pi}\mathrm{i}[\phi_7\boldsymbol{n}(\theta) + \tfrac{1}{4}\phi_I(1+\cos(\theta-\beta))(\boldsymbol{n}(\beta)+\boldsymbol{n}(\theta))]\psi^{1*}\mathrm{d}\theta. \tag{C 25}$$

Consider now $I(\phi^0,\psi^1)$. By introducing (C 9)–(C 14) and (C 18)–(C 20) into (C 21), we obtain

$$I(\phi^0,\psi^1) = \mathrm{Re}\int_{S(R)} K[\phi_7\boldsymbol{n}(\theta) + \tfrac{1}{4}\phi_I(1+\cos(\theta-\beta))(\boldsymbol{n}(\beta)+\boldsymbol{n}(\theta))]\psi^{1*}\mathrm{d}S - \mathrm{Re}\int_{S(R)} R^{-1}\mathrm{i}[\phi_7\psi^{1*}\boldsymbol{n}(\theta) + \tfrac{1}{2}(\phi_7\psi^{1*}_\theta - \phi_{7,\theta}\psi^{1*})\boldsymbol{t}(\theta)]\mathrm{d}S$$

$$-\frac{1}{8}\mathrm{Re}\int_{S(R)} R^{-1}\mathrm{i}\phi_I\psi^{1*}[\boldsymbol{n}(\theta)+\boldsymbol{n}(\beta)+(1+\cos(\theta-\beta))\boldsymbol{n}(\theta)]\mathrm{d}S$$
$$-\frac{1}{4}\mathrm{Re}\int_{S(R)} R^{-1}\mathrm{i}\phi_I\psi_\theta^{1*}[1+\cos(\theta-\beta)]\boldsymbol{t}(\theta)\mathrm{d}S. \qquad (\mathrm{C}\,26)$$

By then carrying out the vertical integration we find

$$I(\phi^0,\psi^1)=\tfrac{1}{2}\mathrm{Re}\int_{C(R)}[\phi_7\boldsymbol{n}(\theta)+\tfrac{1}{4}\phi_I(1+\cos(\theta-\beta))(\boldsymbol{n}(\beta)+\boldsymbol{n}(\theta))]\psi^{1*}R\mathrm{d}\theta$$
$$-\frac{1}{2K}\mathrm{Re}\int_{C(R)}\mathrm{i}[\phi_7\psi^{1*}\boldsymbol{n}(\theta)+\tfrac{1}{2}(\phi_7\psi_\theta^{1*}-\phi_{7,\theta}\psi^{1*})\boldsymbol{t}(\theta)]\mathrm{d}\theta$$
$$-\frac{1}{16K}\mathrm{Re}\int_{C(R)}\mathrm{i}\phi_I\psi^{1*}[\boldsymbol{n}(\theta)+\boldsymbol{n}(\beta)+(1+\cos(\theta-\beta))\boldsymbol{n}(\theta)]\mathrm{d}\theta$$
$$-\frac{1}{8K}\mathrm{Re}\int_{C(R)}\mathrm{i}\phi_I\psi_\theta^{1*}(1+\cos(\theta-\beta))\boldsymbol{t}(\theta)\mathrm{d}\theta. \qquad (\mathrm{C}\,27)$$

Consider finally the part of (C 1) given by

$$I_3=-\frac{1}{2}(\boldsymbol{i}_i\frac{\partial}{\partial\beta}-\boldsymbol{k}\times\boldsymbol{i}_i)\int_{C(R)}x_i\mathrm{Im}(\phi^0\phi_n^{0*})R\mathrm{d}\theta. \qquad (\mathrm{C}\,28)$$

By partial integration we obtain

$$I_3=-\frac{1}{2}\int_{C(R)}\boldsymbol{n}(\theta)\left(\frac{\partial}{\partial\beta}+\frac{\partial}{\partial\theta}\right)\mathrm{Im}(\phi^0\phi_n^{0*})R^2\mathrm{d}\theta. \qquad (\mathrm{C}\,29)$$

By then carrying out the differentiation with respect to the β and θ variables, we find

$$I_3=-\frac{1}{2}\mathrm{Re}\int_{C(R)}[\phi_7\boldsymbol{n}(\theta)+\tfrac{1}{4}\phi_I(1+\cos(\theta-\beta))(\boldsymbol{n}(\theta)+\boldsymbol{n}(\beta))]\psi^{1*}R\mathrm{d}\theta$$
$$-\frac{1}{8}\mathrm{Re}\int_{C(R)}(\boldsymbol{n}(\theta)-\boldsymbol{n}(\beta))(1+\cos(\theta-\beta))\phi_I\psi^{1*}R\mathrm{d}\theta$$
$$-\frac{1}{8K}\mathrm{Re}\int_{C(R)}\mathrm{i}\phi_I\psi^{1*}\boldsymbol{n}(\theta)\mathrm{d}\theta. \qquad (\mathrm{C}\,30)$$

Now we have that $(\boldsymbol{n}(\theta)-\boldsymbol{n}(\beta))(1+\cos(\theta-\beta))=(\boldsymbol{t}(\theta)+\boldsymbol{t}(\beta))\sin(\theta-\beta)$. Furthermore, $KR\sin(\theta-\beta)\mathrm{e}^{\mathrm{i}KR(1-\cos(\theta-\beta))}=-\mathrm{i}(\mathrm{d}/\mathrm{d}\theta)\mathrm{e}^{\mathrm{i}KR(1-\cos(\theta-\beta))}$. This means that

$$-\frac{1}{8}\mathrm{Re}\int_{C(R)}(\boldsymbol{n}(\theta)-\boldsymbol{n}(\beta))(1+\cos(\theta-\beta))\phi_I\psi^{1*}R\mathrm{d}\theta$$
$$=-\frac{1}{8}\mathrm{Re}\int_{C(R)}(\boldsymbol{n}(\theta)-\boldsymbol{n}(\beta))(1+\cos(\theta-\beta))$$
$$\times 2KR^{3/2}(H_\beta^{0*}+H_\theta^{0*})\mathrm{e}^{\mathrm{i}KR(1-\cos(\theta-\beta))}\mathrm{d}\theta(1+O((KR)^{-1}))$$
$$=-\frac{1}{8K}\mathrm{Re}\int_{C(R)}\mathrm{i}\phi_I\psi_\theta^{1*}(\boldsymbol{t}(\theta)+\boldsymbol{t}(\beta))\mathrm{d}\theta-\frac{1}{8K}\mathrm{Re}\int_{C(R)}\mathrm{i}\phi_I\psi^{1*}\boldsymbol{n}(\theta)\mathrm{d}\theta \qquad (\mathrm{C}\,31)$$

where partial integration is applied and we have neglected terms of $O((KR)^{-1})$.

Thus, I_3 is given by

$$I_3=-\frac{1}{2}\mathrm{Re}\int_{C(R)}[\phi_7\boldsymbol{n}(\theta)+\tfrac{1}{4}\phi_I(1+\cos(\theta-\beta))(\boldsymbol{n}(\theta)+\boldsymbol{n}(\beta))]\psi^{1*}R\mathrm{d}\theta$$
$$-\frac{1}{8K}\mathrm{Re}\int_{C(R)}\mathrm{i}\phi_I\psi_\theta^{1*}[\boldsymbol{t}(\theta)+\boldsymbol{t}(\beta)]\mathrm{d}\theta. \qquad (\mathrm{C}\,32)$$

By then summing I_1, $I(\phi^0, \psi^1)$, and I_3, we obtain

$$I_1 + I(\phi^0,\psi^1) + I_3$$
$$= -\mathrm{Re}\left(\int_0^{2\pi} \mathrm{i}H^0(H^0_{\beta\theta} + H^0_{\theta\theta})^* \boldsymbol{t}(\theta)\mathrm{d}\theta + \mathrm{i}\left(\frac{2\pi}{K}\right)^{1/2} \mathrm{e}^{\mathrm{i}\pi/4}(H^{0*}_{\beta\theta}(\beta) + H^{0*}_{\theta\theta}(\beta))\boldsymbol{t}(\beta)\right) \quad \text{(C 33)}$$

where the method of stationary phase is applied. The final expression for $B_{16}\boldsymbol{i} + B_{26}\boldsymbol{j}$ then becomes

$$\frac{B_{16}\boldsymbol{i} + B_{26}\boldsymbol{j}}{\rho g A^2} = \frac{1}{2}\left(\boldsymbol{i}_i\frac{\partial}{\partial\beta} - \boldsymbol{k}\times\boldsymbol{i}_i\right)\int_{S_F}(\chi_i - x_i)\mathrm{Im}(\phi^0\phi^{0*}_{zz})\mathrm{d}S$$
$$+\frac{1}{2}\mathrm{Re}\int_0^{2\pi}\left[H^0(\theta)H^{1*}(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} - 2\mathrm{i}H^0(\theta)(H^{0*}_{\beta\theta}(\theta) + H^{0*}_{\theta\theta}(\theta))\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right]\mathrm{d}\theta$$
$$+\frac{1}{2}\mathrm{Re}\left\{\left(\frac{2\pi}{K}\right)^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}\left[H^{1*}(\beta)\begin{pmatrix}\cos\beta\\ \sin\beta\end{pmatrix} - 2\mathrm{i}(H^{0*}_{\beta\theta}(\beta) + H^{0*}_{\theta\theta}(\beta))\begin{pmatrix}-\sin\beta\\ \cos\beta\end{pmatrix}\right]\right\}. \quad \text{(C 34)}$$

REFERENCES

EMMERHOFF, O. J. 1994 The slow drift motions of offshore structures. PhD Thesis, Massachusetts Institute of Technology.

EMMERHOFF, O. J. & SCLAVOUNOS, P. D. 1992 The slow-drift motion of arrays of vertical cylinders. *J. Fluid Mech.* **242**, 31–50.

EMMERHOFF, O. J. & SCLAVOUNOS, P. D. 1993 The slow-drift yaw-motion of offshore structures. *8th Intl Workshop on Water Waves and Floating Bodies, St. John's* (ed. J. Pawlowski). IMD, Newfoundland, Canada.

FALTINSEN, O. M. 1990 *Sea Loads on Ships and Offshore Structures.* Cambridge University Press.

FINNE, S. & GRUE, J. 1995 Analysis of the forces and the responses of floating bodies with a slow yaw-motion. *10th Intl Workshop on Water Waves and Floating Bodies, Oxford* (ed. R. Eatock Taylor). Dept. of Engng Sciences, University of Oxford, UK.

GRUE, J. & BIBERG, D. 1993 Wave forces on marine structures with small speed in water of restricted depth. *Appl. Ocean Res.* **15**, 121–135.

GRUE, J. & PALM, E. 1985 Wave radiation and wave diffraction from a submerged body in a uniform current. *J. Fluid Mech.* **151**, 257–278.

GRUE, J. & PALM, E. 1986 The influence of a uniform current on slowly varying forces and displacements. *Appl. Ocean Res.* **8**, 232–239.

GRUE, J. & PALM, E. 1993 The mean drift force and yaw moment on marine structures in waves and current. *J. Fluid Mech.* **250**, 121–142.

GRUE, J. & PALM, E. 1994 A boundary element method for predicting wave forces on marine bodies with slow yaw-motion. *Proc. 7th Intl Conf. Behaviour of Offshore Structures (BOSS '94), MIT,* (ed. C. Chryssostomidis), Vol. 2. Pergamon.

HUIJSMANS, R. H. M. 1986 Wave drift forces in current. *16th Symp. on Naval Hydrodyn., Berkeley.* National Academy Press, Washington DC.

HUIJSMANS, R. H. M. & HERMANS, A. J. 1985 A fast algorithm for computation of 3-D ship motions at moderate forward speed. *4th Intl Conf. on Numerical Ship Hydrodynamics* (ed. J. H. McCarthy). National Academy of Science Press, Washington.

NEWMAN, J. N. 1967 The drift force and moment on ships in waves. *J. Ship Res.* **11**, 51–60.

NEWMAN, J. N. 1985 Algoritms for the free-surface Green function. *J. Engng Maths* **19**, 57–67.

NEWMAN, J. N. 1993 Wave-drift damping of floating bodies. *J. Fluid Mech.* **249**, 241–259.

NOSSEN, J., GRUE, J. & PALM, E. 1991 Wave forces on three-dimensional floating bodies with small forward speed. *J. Fluid Mech.* **227**, 135–160.

SCLAVOUNOS, P. D. 1994 Slow-drift oscillation of compliant floating platforms. *Proc. 7th Intl Conf. Behaviour of Offshore Structures (BOSS '94), MIT* (ed. C. Chryssostomidis), Vol. 2. Pergamon.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. *Handbuch der Physik IX.*

WICHERS, J. E. W. & SLUIJS, M. F. VAN 1979 The influence of waves on the low frequency hydrodynamic coefficients of moored vessels. *Proc. Offshore Technology Conf., Houston, OTC 3625.*

ZHAO, R. & FALTINSEN, O. M. 1989 Interaction between current, waves and marine structures. *5th Intl Conf. on Num. Ship Hydrodynamics, Hiroshima* (ed. K. Mori), National Academy Press, Washington.

ZHAO, R., FALTINSEN, O. M., KROKSTAD, J. R. & AANESLAND, V. 1988 Wave-current interaction effects on large-volume structures. *Proc. 5th Intl Conf. Behaviour of Offshore Structures (BOSS '88), Trondheim,* (ed. T. Moan, N. Janby & O. M. Faltinsen), Vol. 2. Tapir Publishers.